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A Regularity Theory for Fractional Harmonic Maps

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Summary

Our purpose is to define, and develop a regularity theory for, Intrinsic Minimising Fractional Harmonic Maps from Euclidean space into smooth compact Riemannian manifolds for fractional powers strictly between zero and one. Our aims are motivated by the theory for Intrinsic Semi-Harmonic Maps, corresponding to the power one-half, developed by Moser [34].

Our definition and methodology are based on an extension method used for the analysis of real valued fractional harmonic functions. We define and derive regularity properties of Fractional Harmonic Maps by regarding their domain as part of the boundary of a half-space, equipped with a Riemannian metric which degenerates or becomes singular on the boundary, and considering the regularity of their extensions to this half-space.

We show that Fractional Harmonic Maps, and their first order derivatives, are locally Hölder continuous away from a set with Hausdorff dimension depending on the dimension of the domain and the fractional power in question. We achieve this by establishing the corresponding partial regularity of extensions of Fractional Harmonic Maps which minimise the Dirichlet energy on the half-space.

To prove local Hölder continuity, we develop several results in the spirit of the regularity theory for harmonic maps. We combine a monotonicity formula with the construction of comparison maps, scaling in the Poincaré inequality and results from the theory of harmonic maps, to prove energy decay sufficient for the application of a modified decay lemma of Morrey.

Using the Hölder continuity of minimisers, we prove a bound for the essential supremum of their gradient. Then we consider the derivatives in directions tangential to the boundary of the half-space; we establish the existence of their gradients using difference quotients. A Caccioppoli-type inequality and scaling in the Poincaré inequality then imply decay estimates sufficient for the application of the modified decay lemma to these derivatives.

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Contents

Chapter 1

Introduction

The aim of this thesis is to define and analyse the regularity properties of fractional harmonic maps between Riemannian manifolds. To motivate this objective and outline our approach we begin with a discussion of fractional harmonic functions.

Fractional harmonic functions appear in a wide range of contexts; quantum mechanics, optimisation and mathematical finance, minimal surfaces and crystal dislocations to name a few [14], and have been extensively studied in the literature. There are several equivalent ways to define the fractional laplacian of a function $u : \mathbb{R}^m \to \mathbb{R}$. Provided u is sufficiently regular we have, for example,

$$(-\Delta)^s u(x) = C(m,s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^m \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{2s+m}} \mathrm{d}y,$$

where $s \in (0,1)$, $m \in \mathbb{N}$ $\varepsilon > 0$ and $B_\varepsilon(x)$ is the ball in $\mathbb{R}^m$ with radius ε and centre x. These operators are naturally associated to the Sobolev spaces $H^s(\mathbb{R}^m)$, see [14], whose norm consists of the L^2 norm of u plus a suitable semi-norm. The square of the semi-norm serves as an energy for $u \in H^s(\mathbb{R}^m)$ and this energy may be expressed as a constant multiple of

$$||(-\Delta)^{\frac{s}{2}} u||^2_{L^2(\mathbb{R}^m)}.$$

In contrast to the Laplace operator, fractional Laplace operators are non-local; the value of $(-\Delta)^s u(x)$ depends upon the behaviour of u on the whole of its domain, not just in a neighbourhood of x. One might therefore expect that techniques, such as localisation or comparison principles, used to study solutions of Laplace's equation $\Delta u = 0$, may not be applicable in the study of solutions to the fractional counterpart of this equation, namely $(-\Delta)^s u = 0$.

Such an equation arises, for instance, as the Euler-Lagrange equation for the energy $||(-\Delta)^{\frac{s}{2}}u||^2_{L^2(\mathbb{R}^m)}$. However, using a result of Caffarelli and Silvestre [5], one may recast this, seemingly non-local, problem as a variant of the problem for Laplace's equation by considering the extension of u to a half-space with $\mathbb{R}^m$ as the boundary.

Let $\mathbb{R}^{m+1}_+ = \mathbb{R}^m \times (0,\infty)$ and $\beta \in (-1,1)$. The result of Caffarelli and Silvestre [5] states that, for given boundary data $u : \mathbb{R}^m \to \mathbb{R}$, solutions $v : \mathbb{R}^{m+1}_+ \to \mathbb{R}$ of the Dirichlet problem:

$$\operatorname{div}(x^\beta_{m+1}\nabla v) = 0 \text{ in } \mathbb{R}^{m+1}_+ \qquad \text{and} \qquad Tv = u \tag{1.0.1}$$

satisfy $(-\Delta)^{\frac{1-\beta}{2}}u = \partial^\beta_{m+1}v := -\lim_{x_{m+1}\to 0^+} x^\beta_{m+1}\partial_{m+1}v$, where T is a suitable trace operator with respect to $\partial\mathbb{R}^{m+1}_+$. We observe that $\operatorname{div}(x^\beta_{m+1}\nabla v) = 0$ is the Euler-Lagrange equation for the Dirichlet energy $E^\beta(v) = \frac{1}{2}\int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x$, defined, for example, on $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$, the homogeneous Sobolev space with $(E^\beta)^{\frac{1}{2}}$ as the norm. Hence, if v is a critical point of E^β with $Tv = u$, then v is a solution of (1.0.1). If we wish to study fractional harmonic functions, that is, maps with $(-\Delta)^s u = 0$, then the aforementioned theory suggests that we consider minimisers of E^β with $Tv = u$ such that $\partial^\beta_{m+1}v = 0$.

There are two possible approaches which lead to a generalisation of the notion of fractional harmonic functions to fractional harmonic maps between Riemannian manifolds. To facilitate their comparison, we give a reformulation of the above variational problem so as to remove any explicit mention of the fractional Laplace operators. One method of proof for Caffarelli and Silvestre's result [5] is to show that $C||(-\Delta)^{\frac{s}{2}}u||^2_{L^2(\mathbb{R}^m)} = E^\beta(v)$ where $s = \frac{1-\beta}{2}$, v is an extension of u satisfying (1.0.1) and C depends on m and s. Consequently, the semi-norm on H^s satisfies

$$C||(-\Delta)^{\frac{s}{2}}u||^2_{L^2(\mathbb{R}^m)} = \inf\{E^\beta(v) : Tv = u, v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)\}.$$

Furthermore, the discussion above yields Dirichlet to Neumann map $u \mapsto \partial^\beta_{m+1}v$ which is identified with the fractional Laplace operator of order $\frac{1-\beta}{2}$. If instead we consider (1.0.1) for $u : \mathcal{O} \to \mathbb{R}$ where $\mathcal{O} \subsetneq \mathbb{R}^m$ is open, then we still obtain a Dirichlet to Neumann map for the problem (1.0.1), but we may no longer identify this with a fractional Laplace operator. We can thus express the conclusions of Caffarelli and Silvestre in the following way: the first variation of the energy, given by $\inf\{E^\beta(v) : Tv = u\}$ on H^s, is the Dirichlet to Neumann map $u \mapsto \partial^\beta_{m+1}v$. As a result, we may study fractional harmonic functions u by considering v which minimise E^β, satisfy $Tv = u$ and send $u \mapsto 0$ via the aforementioned Dirichlet to Neumann map.

This observation is key; it provides one of the foundational ideas for the work conducted in this thesis. Using the aforementioned Dirichlet to Neumann map, we can obtain regularity results for solutions of $(-\Delta)^{\frac{1-\beta}{2}} u = 0$ by examining the corresponding properties of an extension $v : \mathbb{R}^{m+1}_+ \to \mathbb{R}$, as described above, near the boundary $\partial\mathbb{R}^{m+1}_+$. In particular, we may use many of the methods from the theory of second order elliptic partial differential equations to analyse this extension. If v has a continuous, or even smooth (with respect to the variables $x_1, \dots, x_m$) extension to $\mathbb{R}^m \times \{0\}$, then u is continuous or smooth respectively. As a consequence of the results of [7], a map $u : \mathcal{O} \to \mathbb{R}$, whose extension v satisfies $\operatorname{div}(x^\beta_{m+1} \nabla v) = 0$ in $\mathbb{R}^{m+1}_+$ and $\partial^\beta_{m+1} v = 0$ in $\mathcal{O}$, is indeed smooth.

With this formulation in mind, we now discuss the definition of fractional harmonic maps between manifolds. We first consider the situation corresponding to the equation $(-\Delta)^{\frac{1}{2}} u = 0$. Da Lio and Rivière [12] first considered $\frac{1}{2}$-harmonic maps into the round unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ centred at the origin. These are critical points u of the energy $||(-\Delta)^{\frac{1}{4}} u||^2_{L^2(\mathbb{R}^m)}$ in $\dot{H}^{\frac{1}{2}}$ under the constraint that u takes values in $\mathbb{S}^{n-1}$ almost everywhere. In other words, they considered critical points of the functional

$$L(u) = \inf\{E^0(v) : Tv = u, v \in \dot{W}^{1,2}(\mathbb{R}^{m+1}_+; \mathbb{R}^n), u(x) \in \mathbb{S}^{n-1} \text{ for a.e } x \in \mathbb{R}^m\},$$

where $\mathbb{R}^m$ is identified with $\partial\mathbb{R}^{m+1}_+$, which are defined as $\frac{1}{2}$-harmonic maps into $\mathbb{S}^{n-1}$. The motivation for studying critical points of this functional comes from the physical relevence of $\frac{1}{2}$-harmonic maps. As noted in [12], they appear, for instance, in the asymptotic limit of equations in phase-field theory for reaction-diffusion. There is also a geometric reason for studying such a functional; when $m = 1$ the functional is invariant under the trace of conformal transformations of $\mathbb{R}^2_+$.

Moser [34] observed that the definition of L is not intrinsic, meaning that it depends on the choice of embedding of $\mathbb{S}^{n-1}$ into an ambient space. He proposed a modification of L which removes such dependence, and also considers general, compact target manifolds. Suppose N is a smooth compact manifold which, due to the embedding theorem of Nash [36], we may assume is isometrically embedded in $\mathbb{R}^n$ for some n. Define

$$I(u) = \inf\{E^0(v) : Tv = u, v \in \dot{W}^{1,2}(\mathbb{R}^{m+1}_+; \mathbb{R}^n), v(x) \in N \text{ for a.e } x \in \mathbb{R}^{m+1}_+\}. \tag{1.0.2}$$

The critical points of I are called intrinsic semi-harmonic maps. When $m = 1$, this functional is conformally invariant in the same sense as L is, so the geometric motivation for $\frac{1}{2}$-harmonic maps also serves as a reason to study intrinsic semi-harmonic maps in this case. Moser, also considered the functional I when the domain of u is restricted to open $\mathcal{O} \subsetneq \mathbb{R}^m$.

There are two potential methods for calculating the Euler Lagrange equation for L. It is possible to directly calculate the equation, owing to the fact that the energy in question is defined in terms of a differential operator, by considering variations of the energy in H^s which respect the constraint $u \in \mathbb{S}^{n-1}$. This is the approach taken by Da Lio and Rivière [12]. Alternatively, we could try to connect the Dirichlet to Neumann map, similar to that obtained from the Dirichlet problem (1.0.1), to the first variation of L. If successful, we would then obtain an identification of this map with the differential operator obtained in the explicit calculation. We note that any v for which $L(u) = E^0(v)$ satisfies $\Delta v = 0$ in $\mathbb{R}^{m+1}_+$ and is thus smooth in $\mathbb{R}^{m+1}_+$.

In contrast, as there is not an obvious a-priori choice of differential operator corresponding to I, Moser [34] used the latter interpretation of the first variation of I as a Dirichlet to Neumann map. Thus intrinsic semi-harmonic maps u have an extension $v \in \dot{W}^{1,2}(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ with $v(x) \in N$ for almost every x. The map v minimises E^0 among all maps with values in N and trace equal to u and satisfies $v \mapsto 0$ under the Dirichlet to Neumann map. In this case, any v for which $I(u) = E^0(v)$ is a critical point of the energy. However, the constraint $v \in N$ must now be taken into account when calculating the Euler-Lagrange equation for E^0. To facilitate subsequent discussions of these equations we now recall the definition of a harmonic map between Riemannian manifolds.

Let M be a Riemannian manifold, of dimension $m+1$, with metric g and let N be a smooth, compact Riemannian manifold isometrically embedded in $\mathbb{R}^n$ for some $n \in \mathbb{N}$. We are interested in defining the energy of maps in

$$W^{1,2}(M; N) := \{v \in W^{1,2}(M; N) : v(x) \in N \text{ for almost every } x \in M\}.$$

The energy density of $v \in W^{1,2}(M; N)$ is given by $e(v) = \sum_{i,j} g^{ij} \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_j} \right\rangle$ where g_{ij} are the components of the matrix representing g in coordinates and g^{ij} are the components of its inverse. The energy of v is defined as

$$E_g(v) = \int_M e(v) \mathrm{d}vol_M. \tag{1.0.3}$$

If $x_1, \ldots, x_{m+1}$ are coordinates on M then

$$E_g(v) = \int_M \sum_{i,j} g^{ij} \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_j} \right\rangle \sqrt{\det(g)}\mathrm{d}x$$

where $\det(g)$ is the determinant of the matrix representing g in the given coordinates. The Euler-Lagrange equations for E_g are

$$\Delta_g v + \sum_{i,j} g^{ij} A(v) \left(\frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_j} \right) = 0 \tag{1.0.4}$$

where A is the second fundamental form of N, a section of $T^*N \otimes T^*N \otimes (TN)^\perp$, and $\Delta_g v$ is the Laplace-Beltrami operator. For the sake of consistency with the usual Laplace operator, we use the convention that, in coordinates, this operator is given by $\Delta_g v = \sum_{i,j} \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{\det(g)} \frac{\partial}{\partial x_j} v \right)$. Critical points of E_g with respect to the dependent variable (weakly) satisfy (1.0.4) and are called *(weakly) harmonic maps with respect to g*. If the choice of g is implicit from the context or we are talking about harmonic maps corresponding to different metrics, we simply refer to (weakly) harmonic maps.

With the above notion of harmonic map in mind, we may formulate the discussion of intrinsic semi-harmonic maps as follows. Any v with $E^0(v) = I(u)$ for some u satisfies

$$\Delta v + A(v)(\nabla v, \nabla v) = 0 \tag{1.0.5}$$

in $\mathbb{R}^{m+1}_+$, where $A(v)(\nabla v, \nabla v) = \sum_i A(v) \left(\frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)$ and Δ is the usual Laplace operator on Euclidean space. Intrinsic semi-harmonic maps may thus be regarded as maps u which have a harmonic extension v to $\mathbb{R}^{m+1}_+$ satisfying (1.0.5), which minimise E^0 among all maps with trace u and which are sent to 0 by the Dirichlet to Neumann map that arises as the first variation of I.

Da Lio and Rivière considered the regularity of $\frac{1}{2}$-harmonic maps when $m = 1$ and showed that such maps are smooth. They consider the Euler Lagrange equations for L explicitly, the underlying idea being to re-formulate these equations in such a way as to take advantage of compensation phenomena: in this case these are gains in regularity after re-writing the equations using the geometric structure of the sphere. They succeed in writing the equations in a form which is sufficient to deduce decay estimates for the energy which imply continuity of solutions, from which they deduce their smoothness. They extended these methods to $\frac{1}{2}$-harmonic maps u into general C^2 target manifolds N [11]. The idea is still to take advantage of gains in regularity observed via a re-writing of the

Euler-Lagrange equation; the application of a well chosen rotation along u gives the desired structure and it is then possible to prove suitable decay of the energy to conclude continuity and higher regularity.

Moser [34] instead examined the regularity of the extension to $\mathbb{R}^{m+1}_+$ of intrinsic semi-harmonic maps. In particular, he considered the regularity of the extensions of maps u which are critical points of I with respect to both the dependent and independent variable. He showed that if u is such a critical point and v its extension, then v is a critical point of E^0 with respect to variations of the dependent and independent variable; in other words, v is a *stationary harmonic map*. The fact that u is sent to 0 by the first variation of I may be regarded as a 0 Neumann boundary condition for v. Consequently, the even reflection of v in $\partial\mathbb{R}^{m+1}_+$ gives a stationary harmonic map from $\mathbb{R}^{m+1}$ into N. The regularity theory for such maps is known; a stationary harmonic map $v : \mathbb{R}^{m+1} \to N$ is smooth away from a set of points of vanishing $m-1$ dimensional Hausdorff measure when $m \geq 2$ [3] and when $m = 1$ such a v is smooth [26]. The map u then inherits its regularity from an extension v. Moser also showed that if the domain of u is $\mathbb{R}^m$ then u must be constant. He furthermore examined critical points of I with the following constraint: let Γ be a smooth closed submanifold of N and suppose $u(x) \in \Gamma$ for almost every x. Once such critical points are defined appropriately, their regularity is the same as for the unconstrained problem. Moreover, the results may essentially be regarded as generalisations of the theory of Da Lio and Rivière.

So far, we have only discussed fractional harmonic maps corresponding to the power $\frac{1}{2}$. However, Da Lio and Schikkora [10, 42, 13, 43] have also considered the regularity properties of extrinsic fractional harmonic maps for other powers as well as on domains of dimension $m > 1$ which are not necessarily the whole of $\mathbb{R}^m$. The idea in each case is still to take advantage of compensation phenomena to obtain the resulting regularity. Da Lio [10] proved full regularity of $\frac{m}{2}$ harmonic maps $u : \mathbb{R}^m \to N$ whenever m is odd and N is smooth and compact without boundary. Schikkora [42] proved that $\frac{m}{2}$ harmonic maps from a domain into $\mathbb{S}^{n-1}$ are Hölder continuous. Da Lio and Schikkora [13] showed that $\frac{m}{p}$ harmonic maps into $\mathbb{S}^{n-1}$ or, more explicitly, critical points of $\int_{\mathbb{R}^m} |(-\Delta)^{\frac{\alpha}{2}} u|^p \mathrm{d}x$ where $p = \frac{m}{\alpha} \in (1, \infty)$, are Hölder continuous. Schikkora has also considered the techniques used to obtain regularity for fractional harmonic maps in generality [43].

To our knowledge, there are currently no results regarding the regularity of intrinsic fractional harmonic maps for powers other than $\frac{1}{2}$. In this thesis we generalise Moser's approach to a regularity theory for fractional harmonic maps to powers strictly between 0 and 1.

Chapter 2

Weighted Sobolev Spaces and Degenerate Elliptic Equations

We intend to define and study two related families of variational problems, one involving a family of Dirichlet energies and the other, a family of related functionals, both parametrised by $\beta \in (-1, 1)$. One set of problems will be posed for maps defined on the half space $\mathbb{R}^{m+1}_+ = \mathbb{R}^m \times (0, \infty)$ for $m \in \mathbb{N}$ and the other will be posed for maps defined on open subsets of the boundary of $\mathbb{R}^{m+1}_+$. In order to observe the connection we expect between these problems, we must work with function spaces where the boundary values of functions and their derivatives need not vanish. Appropriate choices are homogeneous Sobolev spaces defined with respect to the energies that we will consider later.

The Euler-Lagrange equations for the Dirichlet energies are in the form of degenerate elliptic semi-linear equations. To facilitate the study of such equations, we will need to examine their linear parts in more detail. These have the form

$$\operatorname{div}(x_{m+1}^{\beta} \nabla v) = 0 \tag{2.0.1}$$

on open subsets of $\mathbb{R}^{m+1}_+$, together with either Dirichlet or Neumann-type boundary data depending on the context. When $\beta \in (-1, 1)\backslash\{0\}$, the equations are degenerate elliptic; there are no uniform bounds, either away from zero when $\beta \in (0, 1)$ or away from infinity when $\beta \in (-1, 0)$, for the coefficients of the highest order terms.

Degenerate elliptic equations with certain classes of coefficients may be analysed in a canonically associated weighted Sobolev space. The homogeneous Sobolev spaces we will use to study the aforementioned variational problems are connected, locally, to the weighted Sobolev spaces corresponding to (2.0.1). We will first introduce these Sobolev Spaces and describe the connections between them. Then we discuss relevant properties of solutions to (2.0.1).

2.1 Weighted Sobolev Spaces

Certain classes of degenerate elliptic equations may be studied using weighted Sobolev and Lebesgue spaces. Our purpose here is to introduce the function spaces relevant to the study of (2.0.1). Rather than give a complete overview of these spaces, we present the machinery necessary for our subsequent analysis. When dealing with solutions of (2.0.1) in domains overlapping $\partial\mathbb{R}^{m+1}_+ = \mathbb{R}^m \times \{0\}$ it will often be helpful to reflect the solutions in this hyperplane, to this end we consider function spaces for maps with open domain $\Omega \subset \mathbb{R}^{m+1}$. Furthermore, the maps we consider will be vector-valued with image contained in $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

Let $\mathrm{d}x$ denote the Lebesgue measure on $\mathbb{R}^{m+1}$ for $m \in \mathbb{N}$. Any non-negative, locally integrable function w defines a weighted measure, $\mathrm{d}\mu_w(x)$, with respect to $\mathrm{d}x$ via $\mathrm{d}\mu_w(x) = w(x)\mathrm{d}x$ wherever the function is defined. The function w is said to be a weight and is the Radon-Nikodym derivative of $\mathrm{d}\mu_w(x)$ with respect to $\mathrm{d}x$.

The class of weights relevant to the problems we consider are Muckenhoupt's A_p weights with respect to the Lebesgue measure on $\mathbb{R}^{m+1}$ for $p \in [1, \infty)$ [35]. Muckenhoupt states the condition to be in A_p in terms of cubes but we may equivalently state it in terms of balls: a weighting w of $\mathrm{d}x$ is said to be in $A_p(\mathrm{d}x)$, or A_p for short, if

$$\sup_{B \subset \mathbb{R}^{m+1}} \left(\frac{1}{\int_B \mathrm{d}x} \int_B w \mathrm{d}x \right) \left(\frac{1}{\int_B \mathrm{d}x} \int_B w^{\frac{-1}{p-1}} \mathrm{d}x \right)^{p-1} \leq C$$

for a positive constant C. The significance of this condition is that it is both necessary and sufficient to ensure the boundedness of the Hardy-Littlewood maximal function, defined with respect to $\mathrm{d}x$, from the Lebesgue space $L^p(\mu_w)$ into itself. This was first proved for weightings of the Lebesgue measure in $\mathbb{R}^{m+1}$ by Muckenhoupt [35] and has been extended to a more general setting by Calderon [8]. Our interest in the A_p condition is motivated by the consequences of the theory of these weights with regard to Sobolev spaces and degenerate elliptic equations; there are a number of similarities to the theory of elliptic equations studied in the usual Sobolev spaces which we will take advantage of.

The Lebesgue and Sobolev spaces best suited to the study of (2.0.1) are those defined with respect to the weights x_{m+1}^{β} where $\beta \in (-1, 1)$. Whenever we consider (2.0.1) in a domain in $\mathbb{R}_+^{m+1}$ whose boundary intersects $\partial\mathbb{R}_+^{m+1}$, the boundary conditions we attach to the equation will allow us to reflect solutions in this hyperplane, giving rise to the weights $|x_{m+1}|^{\beta}$. Hence we consider these weights on subsets of $\mathbb{R}^{m+1}$.

A calculation verifies that $|x_{m+1}|^{\beta}$ is an A_2 weight on $\mathbb{R}^{m+1}$. Consequently $|x_{m+1}|^{\beta}$ is an A_q weight for all $q \geq 2$ by part 3 of remark 1.2.4 in [47]. We define the β-measures

$$\mathrm{d}\mu_{\beta}(x) = |x_{m+1}|^{\beta}\mathrm{d}x \tag{2.1.1}$$

for $\beta \in (-1, 1)$. It is implicit in the definition of $\mathrm{d}\mu_{\beta}(x)$ that these measures are defined on the same sigma-algebra as the Lebesgue measure and hence define the same collection of measurable functions as $\mathrm{d}x$. Thus we may refer to measurable functions without ambiguity.

Let $p \in [1, \infty)$ and Ω be an arbitrary subset of $\mathbb{R}^{m+1}$. Define the weighted Lebesgue spaces corresponding to $\mathrm{d}\mu_{\beta}$ as

$$L^p_{\beta}(\Omega; \mathbb{R}^n) = \{f : \Omega \to \mathbb{R}^n : f \text{ is measurable}, \int_{\Omega} |f|^p \, \mathrm{d}\mu_{\beta} < \infty\} \tag{2.1.2}$$

and for $p = \infty$ we recall the Lebesgue space

$$L^{\infty}(\Omega; \mathbb{R}^n) = \{f : \Omega \to \mathbb{R}^n : f \text{ is measurable}, \operatorname{ess\,sup}_{\Omega}(f) < \infty\}. \tag{2.1.3}$$

As stated in [9] Theorem 3.4.1, these are Banach spaces with respect to the norm $||f||_{L^p_{\beta}(\Omega;\mathbb{R}^n)} = \left(\int_{\Omega} |f|^p \, \mathrm{d}\mu_{\beta}\right)^{\frac{1}{p}}$. Furthermore, when $p = 2$, this norm is induced by the inner product $\langle f, g\rangle_{L^2_{\beta}(\Omega;\mathbb{R}^n)} = \int_{\Omega} \langle f, g\rangle \, \mathrm{d}\mu_{\beta}$ for $f, g \in L^2_{\beta}(\Omega; \mathbb{R}^n)$ where $\langle f, g\rangle$ is the inner product of f and g in $\mathbb{R}^n$. Thus each $L^2_{\beta}(\Omega; \mathbb{R}^n)$ is a Hilbert space. If $\beta = 0$ then we omit the subscript and let $L^p(\Omega; \mathbb{R}^n)$ denote the Lebesgue space of p-integrable functions on Ω with values in $\mathbb{R}^n$.

There are two possible ways to define Sobolev spaces on an open $\Omega \subset \mathbb{R}^{m+1}$. One method is to take the completion of the set of smooth functions from Ω to $\mathbb{R}^n$ with finite Sobolev norm. The other is to equip the collection of L^p_{β} functions, with first order weak derivatives also in L^p_{β}, with a Sobolev norm. The question is then to what extent do the two definitions agree; if they give rise to different function spaces, we must make a choice as to which is better suited to the study of the equations we consider.

Let $p \in [2, \infty)$ and define

$$W^{1,p}_{\beta}(\Omega; \mathbb{R}^n) = \{v : \Omega \to \mathbb{R}^n : v, \frac{\partial v}{\partial x_i} \in L^p_{\beta}(\Omega; \mathbb{R}^n) \text{ for } i = 1, \ldots, m+1\} \quad (2.1.4)$$

where $\frac{\partial v}{\partial x_i}$ denotes the weak (distributional) partial derivative of v with respect to x_i. Proposition 2.1.2 of [47] guarantees that $W^{1,p}_{\beta}(\Omega; \mathbb{R}^n)$ is a Banach space with respect to the norm

$$||v||_{W^{1,p}_{\beta}(\Omega;\mathbb{R}^n)} = \left(\int_{\Omega} |v|^p \mathrm{d}\mu_{\beta} + \int_{\Omega} |\nabla v|^p \mathrm{d}\mu_{\beta}\right)^{\frac{1}{p}}$$

where ∇v is the weak derivative of $v : \Omega \to \mathbb{R}^n$ and $|\nabla v|^2 = \sum_{i=1}^{m+1} \left|\frac{\partial v}{\partial x_i}\right|^2$. Let $C^{\infty}(\Omega; \mathbb{R}^n)$ denote the space of smooth functions from Ω to $\mathbb{R}^n$. It follows from corollary 2.1.6 [47] that the collection of maps in $C^{\infty}(\Omega; \mathbb{R}^n)$ which satisfy $||v||_{W^{1,p}_{\beta}(\Omega;\mathbb{R}^n)} < \infty$ is dense in $W^{1,2}_{\beta}(\Omega; \mathbb{R}^n)$. Hence, for $p \in [2, \infty)$, the space $W^{1,p}_{\beta}$ coincides with the completion of the set of smooth functions with $||v||_{W^{1,p}_{\beta}(\Omega;\mathbb{R}^n)} < \infty$ and the two approaches we could have used to define $W^{1,p}_{\beta}$ give the same function space. This conclusion also holds for all $p \in (q, \infty)$ where $q = q(\beta) \in (1, 2)$ using Corollary 1.2.17 of [47], but not necessarily all $p \in [1, \infty)$. However, throughout this thesis we only consider $W^{1,p}_{\beta}$ when $\beta \neq 0$ for $p \geq 2$. The norm on $W^{1,2}_{\beta}(\Omega; \mathbb{R}^n)$ is induced by the inner product $\langle v, w\rangle_{W^{1,2}_{\beta}(\Omega;\mathbb{R}^n)} = \int_{\Omega} \langle v, w\rangle \, \mathrm{d}\mu_{\beta} + \int_{\Omega} \langle \nabla u, \nabla v\rangle \, \mathrm{d}\mu_{\beta}$ for $v, w \in L^2_{\beta}(\Omega; \mathbb{R}^n)$ where we use the notation $\langle \nabla v, \nabla w\rangle = \sum_{i=1}^{m+1} \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_i}\right\rangle$ to denote the inner product of ∇v and ∇w in $\mathbb{R}^{(m+1)\times n}$. Hence $W^{1,2}_{\beta}(\Omega; \mathbb{R}^n)$ is a Hilbert space and thus reflexive. When $\beta = 0$ we omit the subscript 0 and write $W^{1,p}(\Omega; \mathbb{R}^n)$ for the Sobolev space of p-integrable functions with p-integrable weak first derivatives.

It is worth noting that, for every $\beta \in (-1, 1)\backslash\{0\}$, approximation by smooth functions in $L^p_{\beta}(\Omega; \mathbb{R}^n)$ and $W^{1,p}_{\beta}(\Omega; \mathbb{R}^n)$ works in the same way as for the unweighted spaces $L^p(\Omega; \mathbb{R}^n)$ and $W^{1,p}(\Omega; \mathbb{R}^n)$ whenever $p \geq 2$. The details of this process are given in Theorem 2.1.4 and Corollary 2.1.5 in [47].

In order to find solutions to the Dirichlet problem for (2.0.1) we will need subspaces of the weighted Sobolev and Lebesgue spaces with vanishing boundary values. Define $W^{1,p}_{\beta,0}(\Omega; \mathbb{R}^n)$ as the closure of $C^{\infty}_0(\Omega; \mathbb{R}^n)$ in $W^{1,p}_{\beta}(\Omega; \mathbb{R}^n)$ with respect to $|| \cdot ||_{W^{1,p}_{\beta}(\Omega;\mathbb{R}^n)}$. If $p = 2$ then, as closed subspaces of Hilbert spaces, the $W^{1,2}_{\beta,0}(\Omega; \mathbb{R}^n)$ are also Hilbert spaces.

2.2 Weighted Homogeneous Sobolev Spaces

The Dirichlet energies we intend to study, introduced in detail in the beginning of Chapter 3, are of the form

$$E^{\beta}(v) = \frac{1}{2}\int_{\mathbb{R}^{m+1}_+} |\nabla v|^2 \mathrm{d}\mu_\beta \tag{2.2.1}$$

where $\beta \in (-1,1)$, $\mathrm{d}\mu_\beta(x) = |x_{m+1}|^\beta \mathrm{d}x$ is the measure defined by (2.1.1), $\mathbb{R}^{m+1}_+ = \mathbb{R}^m \times (0,\infty)$ and v is in a function space to be defined here. As stated previously, in order to link the variational problem for the E^β to the related problems for functions defined on open subsets of $\partial\mathbb{R}^{m+1}_+$, we intend to analyse the energies in function spaces where their values and the values of their derivatives on $\partial\mathbb{R}^{m+1}_+$ may be non-zero. An appropriate choice are homogeneous Sobolev spaces defined with respect to the energy; we have control of the energy in these spaces and no extraneous information is contained in the norm.

To construct a suitable Sobolev space for the energies E^β, we consider the completion of a space of smooth functions with respect to norms defined via the energies. Let $m, n \in \mathbb{N}$ and observe that the restriction of functions in $C^\infty_0(\mathbb{R}^{m+1};\mathbb{R}^n)$ to $\mathbb{R}^{m+1}_+$ need not vanish near the boundary of $\mathbb{R}^{m+1}_+$. Define

$$\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n) = \{ \ \phi \mid \phi = f|_{\mathbb{R}^{m+1}_+} \text{ for some } f \in C^\infty_0(\mathbb{R}^{m+1};\mathbb{R}^n)\}.$$

We define a family of norms on $\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$. We write $\frac{\partial}{\partial x_i}$ to denote the partial derivative with respect to the ith variable and ∇ to denote the gradient of a map from $\mathbb{R}^{m+1}_+$ to $\mathbb{R}^n$. Let $\beta \in (-1,1)$ and consider the functional

$$\nu_\beta : \mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n) \to \mathbb{R} : \phi \mapsto \left(\frac{1}{2}\int_{\mathbb{R}^{m+1}_+} |\nabla \phi|^2 \, \mathrm{d}\mu_\beta\right)^{\frac{1}{2}}. \tag{2.2.2}$$

We now show that ν_β is a norm on $\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$.

Lemma 2.2.0.1. *Let $\beta \in (-1,1)$. The function ν_β is a norm on $\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$.*

Proof. If $\phi \equiv 0$ then $\nu_\beta(0) = 0$. Conversely, if $\nu_\beta(\phi) = 0$ then $|\nabla\phi|^2 = 0$ in $\mathbb{R}^{m+1}_+$ and thus ϕ is constant. Since the only constant function in $\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ is the zero function, ϕ must be zero. The fact that $\nu_\beta(c\phi) = |c|\nu_\beta(\phi)$ for all $c \in \mathbb{R}$ follows from the definition of ν_β. Lastly, the fact that the triangle inequality holds for ν_β is a consequence of Minkowski's inequality for the space $L^2_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^{n\times(m+1)})$. $\square$

We are now in a position to define the Sobolev spaces we will use to study the Dirichlet energies.

Definition 2.2.0.1. Let $\beta \in (-1,1)$. The *Weighted Homogeneous Sobolev Space* $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ is the completion of $\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ with respect to the metric induced by ν_β.

By construction, $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ is a Banach space. However, we can show more; the map $\langle\cdot,\cdot\rangle_{\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)} : \mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)\times\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)\to\mathbb{R}$ given by

$$\langle\phi,\psi\rangle_{\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)} = \int_{\mathbb{R}^{m+1}}\langle\nabla\phi,\nabla\psi\rangle \mathrm{d}\mu_\beta$$

for $\phi,\psi\in\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ is well defined as a consequence of Hölder's inequality for maps in $L^2_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^{(m+1)\times n})$. Furthermore, this map is an inner product which induces the norm ν_β on $\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ making it an inner product space. Thus $\langle\cdot,\cdot\rangle_{\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)}$ extends to an inner product on $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$. Since this space is complete by definition it is thus a Hilbert Space and hence reflexive.

2.2.1 Relationship of $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ to Weighted and Unweighted Sobolev Spaces

The elements of $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ are, strictly speaking, equivalence classes of Cauchy sequences and it will be necessary to have more tangible characterisations of these classes in order to solve the partial differential equations we consider.

The $\beta\in(-1,1)$ which parametrise the Dirichlet energies come from the definition of $\mathbb{R}^{m+1}_+$ as a Riemannian manifold as we will see later in the beginning of section 3. When $m=1$, as we will discuss further in section 3.2, the Dirichlet energies all reduce to E^0, the energy for the Lebesgue measure. Thus the problems we consider will be posed for $m\geq 2$. We will show that it is possible to relate $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ to $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ for $\Omega\subset\mathbb{R}^{m+1}_+$ using the Sobolev Embedding Theorem for $W^{1,2}$ when $m\geq 3$ and $\beta\in(-1,1)$. We can obtain the same relation when $m=2$ for $\beta\in(-\frac{1}{3},1)$ using a different method of proof, but we currently do not know if this can be extended to all $\beta\in(-1,1)$ when $m=2$.

Lemma 2.2.1.1. *Let $m\in\mathbb{N}$ with $m\geq 2$ and Ω be an open bounded subset of $\mathbb{R}^{m+1}_+$. If $m=2$ let $\beta\in(-\frac{1}{3},1)$ and if $m\geq 3$ let $\beta\in(-1,1)$. Then there is a bounded linear operator $I:\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)\to W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ which satisfies $If=f|_\Omega$ for every $f\in\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$. Moreover,*

$$||Iv||_{W^{1,2}_\beta(\Omega;\mathbb{R}^n)}\leq C||v||_{\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)} \tag{2.2.3}$$

for every $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ where C is a positive constant depending on m and the diameter of Ω.

Proof. First, suppose that (2.2.3) is true for all $\phi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$, the dense subset of smooth functions in $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$. The operator $\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n) \to W^{1,2}_\beta(\Omega;\mathbb{R}^n) : f \mapsto f|_\Omega$ may be uniquely extended to the operator I described in the statement of the Lemma by taking limits on both sides of (2.2.3) and using the completeness of $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$. Hence we only need show that (2.2.3) holds for all $\phi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$.

We may bound the gradient term in the $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$-norm of ϕ using the $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$-norm of ϕ. To conclude the proof we must show that

$$\int_\Omega |\phi|^2 \mathrm{d}\mu_\beta \leq C \int_{\mathbb{R}^{m+1}_+} |\nabla \phi|^2 \mathrm{d}\mu_\beta \tag{2.2.4}$$

for a constant C as specified in the statement of the lemma. In order to show (2.2.4) we will apply the Sobolev Embedding Theorem (with respect to the Lebesgue measure) on the intersection of hyperplanes which are orthogonal to the $m+1$ axis, with Ω.

For the remainder of the proof, C' denotes a positive constant that only depends on m. Observe that for every $x_{m+1} \in [0,\infty)$ we have $\phi(\cdot, x_{m+1}) \in L^2(\mathbb{R}^m;\mathbb{R}^n)$ and $\nabla'\phi(\cdot,x_{m+1}) \in L^2(\mathbb{R}^m;\mathbb{R}^{mn})$, where ∇' denotes the derivative of ϕ with respect to x' for $(x', x_{m+1}) \in \mathbb{R}^m \times [0,\infty)$. Hence $\phi(\cdot, x_{m+1}) \in W^{1,2}(\mathbb{R}^m;\mathbb{R}^n)$ for every $x_{m+1} \in [0,\infty)$ and we may apply the Sobolev inequality for this space to $\phi(\cdot, x_{m+1})$. The Sobolev exponent in this case is $\frac{2m}{m-2}$ which satisfies $\frac{2m}{m-2} \geq 2$ for every $m \geq 3$. Thus, for every $x_{m+1} \in [0,\infty)$,

$$\left(\int_{\mathbb{R}^m} |\phi(x', x_{m+1})|^{\frac{2m}{m-2}} \mathrm{d}x'\right)^{\frac{m-2}{2m}} \leq C' \left(\int_{\mathbb{R}^m} |\nabla'\phi(x', x_{m+1})|^2 \mathrm{d}x'\right)^{\frac{1}{2}} \tag{2.2.5}$$

for a constant C'.

To see that (2.2.4) holds we will show something stronger: $||\phi||^2_{L^2_\beta(\Omega;\mathbb{R}^m)}$ is bounded in terms of $||\nabla'\phi||^2_{L^2_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^{mn})}$. Define $l(x_{m+1}) = \Omega \cap (\mathbb{R}^m \times \{x_{m+1}\})$ for $x_{m+1} \in [0,\infty)$ and let $a = \inf_\Omega(x_{m+1})$ and $b = \sup_\Omega(x_{m+1})$. Notice that $l(x_{m+1})$ is contained in an m dimensional cube with side length $2\mathrm{diam}(\Omega)$ for every $x_{m+1} \in [0,\infty)$. Hence $\int_{l(x_{m+1})} \mathrm{d}x' \leq 2^m \mathrm{diam}(\Omega)^m$. An application of Fubini's Theorem, followed by an application of Hölder's inequality, with conjugate exponents $\frac{m}{m-2}$

and $\frac{m}{2}$, yields

$$\begin{aligned}
&\int_{\Omega} x_{m+1}^{\beta}|\phi|^2\mathrm{d}x\\
&=\int_a^b x_{m+1}^{\beta}\int_{l(x_{m+1})}|\phi(x',x_{m+1})|^2\mathrm{d}x'\mathrm{d}x_{m+1}\\
&\leq C'\left(\mathrm{diam}(\Omega)\right)^2\int_a^b x_{m+1}^{\beta}\left(\int_{l(x_{m+1})}|\phi(x',x_{m+1})|^{\frac{2m}{m-2}}\mathrm{d}x'\right)^{\frac{m-2}{m}}\mathrm{d}x_{m+1}. \qquad (2.2.6)
\end{aligned}$$

Combining (2.2.6) with (2.2.5) and using Fubini's Theorem once more, we see that

$$\begin{aligned}
\int_{\Omega} x_{m+1}^{\beta}|\phi|^2\mathrm{d}x &\leq C'\left(\mathrm{diam}(\Omega)\right)^2\int_a^b x_{m+1}^{\beta}\left(\int_{l(x_{m+1})}|\phi(x',x_{m+1})|^{\frac{2m}{m-2}}\mathrm{d}x'\right)^{\frac{m-2}{m}}\mathrm{d}x_{m+1}\\
&\leq C'\left(\mathrm{diam}(\Omega)\right)^2\int_a^b x_{m+1}^{\beta}\int_{\mathbb{R}^m}|\nabla'\phi(x',x_{m+1})|^2\mathrm{d}x'\mathrm{d}x_{m+1}\\
&\leq C'\left(\mathrm{diam}(\Omega)\right)^2\int_{\mathbb{R}^{m+1}_+} x_{m+1}^{\beta}|\nabla'\phi|^2\mathrm{d}x. \qquad (2.2.7)
\end{aligned}$$

Hence, using the fact that $\int_{\Omega} x_{m+1}^{\beta}|\nabla\phi|^2\mathrm{d}x \leq \int_{\mathbb{R}^{m+1}_+} x_{m+1}^{\beta}|\nabla\phi|^2\mathrm{d}x$, together with (2.2.6), yields (2.2.4) for $\phi\in\mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ which concludes the proof for $m\geq 3$.

When $m=2$ the preceding method used to establish (2.2.4) is no longer viable; we are working in $\mathbb{R}^3_+$ and are therefore no longer permitted to apply the same Sobolev Embedding Theorem along hyperplanes orthogonal to the 3rd coordinate axis. We have found a substitute for this method which holds when $m=2$ provided $\beta>-\frac{1}{3}$. In this case, Corollary 2 in Section 2.1.7 of [30] implies that for every $\phi\in C_0^{\infty}(\mathbb{R}^3;\mathbb{R}^n)$ we have

$$\left(\int_{\mathbb{R}^3}|x_3|^{3\beta}|\phi|^6\mathrm{d}x\right)^{\frac{1}{3}}\leq C\int_{\mathbb{R}^3}|x_3|^{\beta}|\nabla\phi|^2\mathrm{d}x.$$

By approximation this holds for $\phi\in W^{1,2}_{\beta}(\mathbb{R}^3;\mathbb{R}^n)$. The even reflection of a $\phi\in\mathcal{D}_+(\mathbb{R}^3_+;\mathbb{R}^n)$ in $\partial\mathbb{R}^{m+1}_+$, which we denote $\tilde{\phi}$, is in $W^{1,2}_{\beta}(\mathbb{R}^3;\mathbb{R}^n)$. Hence, we deduce the preceding inequality holds for $\tilde{\phi}$. Furthermore, Hölder's inequality yields

$$\begin{aligned}
\int_{\Omega} x_3^{\beta}|\phi|^2\mathrm{d}x &\leq |\Omega|^{\frac{2}{3}}\left(\int_{\mathbb{R}^3}|x_3|^{3\beta}|\tilde{\phi}|^6\mathrm{d}x\right)^{\frac{1}{3}}\\
&\leq C|\Omega|^{\frac{2}{3}}\int_{\mathbb{R}^3}|x_3|^{\beta}|\nabla\tilde{\phi}|^2\mathrm{d}x = 2C|\Omega|^{\frac{2}{3}}\int_{\mathbb{R}^3_+}x_3^{\beta}|\nabla\phi|^2\mathrm{d}x,
\end{aligned}$$

for every $\phi \in \mathcal{D}_+(\mathbb{R}^3_+; \mathbb{R}^n)$. An approximation argument then yields the statement of the lemma, as in the case $m \geq 3$. $\square$

Remark 2.2.1.1. Henceforth, when stating results for the space $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ we will always assume $m \geq 2$. If $m \geq 3$ then we allow $\beta \in (-1, 1)$ and if $m = 2$ we allow $\beta \in (-\frac{1}{3}, 1)$. In the forthcoming chapters, Chapter 3, Chapter 5 and Chapter 6, we always make these assumptions for m and β. However, we will only usually state the results for $\beta \in (-1, 1)$; if $m = 2$ then we are assuming $\beta \in (-\frac{1}{3}, 1)$.

Corollary 2.2.1.1. *Let $m \in \mathbb{N}$ with $m \geq 3$ and Ω be an open bounded subset of $\mathbb{R}^{m+1}_+$. Furthermore, let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$. Then*

$$\int_\Omega x^\beta_{m+1} |v|^2 \mathrm{d}x \leq C' \left(\mathrm{diam}(\Omega)\right)^2 \int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} |\nabla' v|^2 \mathrm{d}x.$$

Proof. The inequality holds for all $\phi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ as this is (2.2.7) from the proof of Lemma 2.2.1.1. By approximation we deduce the result for a general $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$. $\square$

We can further connect $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ with $W^{1,p}(\Omega; \mathbb{R}^n)$ by examining the relationship between the weighted Sobolev space $W^{1,2}_\beta(\Omega; \mathbb{R}^n)$ and the usual Sobolev spaces $W^{1,p}(\Omega; \mathbb{R}^n)$ for $p \in [1, \infty)$. The proximity of Ω to $\partial\mathbb{R}^{m+1}_+$ is the crucial factor in determining the nature of the connection between these spaces.

Lemma 2.2.1.2. *Let $\beta \in (-1, 1)$ and suppose $\Omega \subset \mathbb{R}^{m+1}_+$ is open, bounded and satisfies $\overline{\Omega} \subset \mathbb{R}^{m+1}_+$. Then $W^{1,2}_\beta(\Omega; \mathbb{R}^n) = W^{1,2}(\Omega; \mathbb{R}^n)$.*

Proof. For such Ω, the norms $|| \cdot ||_{W^{1,2}_\beta(\Omega;\mathbb{R}^n)}$ and $|| \cdot ||_{W^{1,2}(\Omega;\mathbb{R}^n)}$ are equivalent and the statement of the Lemma follows for every $\beta \in (-1, 1)$. $\square$

If $\partial\Omega$ intersects $\partial\mathbb{R}^{m+1}_+$ the situation is different depending on the sign of β. The following two Lemmata summarise the relationship between the weighted and usual Sobolev spaces in this situation.

Lemma 2.2.1.3. *Let $\beta \in (-1, 0]$ and suppose $\Omega \subset \mathbb{R}^{m+1}_+$ is open and bounded. Then $W^{1,2}_\beta(\Omega; \mathbb{R}^n) \subset W^{1,2}(\Omega; \mathbb{R}^n)$.*

Proof. The norm $|| \cdot ||_{W^{1,2}(\Omega;\mathbb{R}^n)}$ is dominated by a constant (depending on Ω and β) multiple of the norm $|| \cdot ||_{W^{1,2}_\beta(\Omega;\mathbb{R}^n)}$ on such Ω. Thus the Lemma is proved. $\square$

Lemma 2.2.1.4. *Let $\beta \in (0, 1)$ and suppose $\Omega \subset \mathbb{R}^{m+1}_+$ is open and bounded. Then $W^{1,2}_\beta(\Omega; \mathbb{R}^n) \subset W^{1,p}(\Omega; \mathbb{R}^n)$ for every $1 \leq p < \frac{2}{1+\beta}$.*

Proof. Let $1 > \beta > 0$ and $1 \leq p < 2$. Then for every measurable $f \in L^2_\beta(\Omega;\mathbb{R}^n)$ we have

$$\begin{aligned}\int_\Omega |f|^p \mathrm{d}x &= \int_\Omega x_{m+1}^{-\frac{p\beta}{2}} x_{m+1}^{\frac{p\beta}{2}} |f|^p \mathrm{d}x \\ &\leq \left(\int_\Omega (x_{m+1}^{-\frac{p\beta}{2}})^q \mathrm{d}x\right)^{\frac{1}{q}} \left(\int_\Omega x_{m+1}^{\beta} |f|^2 \mathrm{d}x\right)^{\frac{p}{2}} \qquad (2.2.8)\end{aligned}$$

where $q = \frac{2}{2-p}$ is the conjugate exponent of $\frac{2}{p}$. The right hand side of (2.2.8) is finite whenever $\left(\int_\Omega (x_{m+1}^{-\frac{p\beta}{2}})^q \mathrm{d}x\right)^{\frac{1}{q}} < \infty$. Since Ω is bounded, we calculate that this is the case as long as

$$1 - \frac{\beta p}{2-p} > 0$$

and rearranging gives the condition $p < \frac{2}{1+\beta}$. It follows from (2.2.8) that if $\beta \in (0,1)$ and $v \in W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ then

$$||v||_{W^{1,p}(\Omega;\mathbb{R}^n)} \leq C||v||_{W^{1,2}_\beta(\Omega;\mathbb{R}^n)}$$

for every $p < \frac{2}{1+\beta}$ where C is a positive constant that depends on Ω and p and hence on β. □

Consequently, in view of Lemmata 2.2.1.1, 2.2.1.2, 2.2.1.3 and 2.2.1.4 we have shown that, provided $m \geq 3$, the restrictions of elements of $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ to open, bounded $\Omega \subset \mathbb{R}^{m+1}_+$ are elements of $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ and hence of $W^{1,p}(\Omega;\mathbb{R}^n)$ for some $1 \leq p \leq 2$ depending on Ω and β.

2.3 Analytical Properties of Weighted Sobolev Spaces

There is general theory of weighted Sobolev Spaces defined via measures satisfying Muckenhoupt's, and even more general, weight conditions. For instance, an axiomatic approach to this theory, tailored to the study of degenerate elliptic equations, is described in Chapter 1 of [25]. A more comprehensive account is provided in Chapter 2 of [47].

Shortly we will prove results required to study the variational problems related to the energies E^β. It is possible, due to the nature of the weight functions x^β_{m+1}, to obtain the necessary results using only the theory of the Sobolev spaces $W^{1,2}$, and not the theory of weighted Sobolev spaces. This approach has the advantage that, for some of the inequalities, we are able to prove estimates with constants independent of $\beta \in (-1,1)$. When considering results on a domain whose boundary intersects $\partial\mathbb{R}^{m+1}_+$, it will suffice to assume the domain is either a half-cube or half-ball with centre in $\partial\mathbb{R}^{m+1}_+$.

2.3.1 Behaviour of Integrals Over Balls and Cubes Under Bi-Lipschitz Maps

Most of the integral estimates we will use are given over balls contained in $\mathbb{R}^{m+1}_+$ or half-balls with centres in $\mathbb{R}^m \times \{0\}$ where $m \in \mathbb{N}$. However, sometimes it may be necessary, or more convenient, to consider integrals over cubes, half-cubes, rectangles or other simple domains instead. We need a means to compare estimates over such domains, to this end we now discuss how integrals with respect to the measures $\mathrm{d}\mu_\beta$ transform under bi-Lipschitz mappings between subsets of $\mathbb{R}^{m+1}$.

Let $\Omega, \hat{\Omega} \subset \mathbb{R}^{m+1}$ be open. Suppose $\Phi : \Omega \to \Phi(\Omega) = \hat{\Omega}$ is a bi-Lipschitz map with inverse $\Phi^{-1} : \hat{\Omega} \to \Omega$. The classical derivatives $\nabla\Phi$ and $\nabla\Phi^{-1}$ exist almost everywhere in their respective domains by Rademacher's Theorem. Let f be a non-negative, integrable function. A change of variables gives

$$\int_{\Phi(\Omega)} x_{m+1}^\beta f \mathrm{d}x = \int_\Omega (\Phi(x))_{m+1}^\beta f(\Phi(x)) |\det(\nabla\Phi(x))| \mathrm{d}x$$

and

$$\int_{\Phi^{-1}(\hat{\Omega})} x_{m+1}^\beta f \mathrm{d}x = \int_\Omega (\Phi^{-1}(x))_{m+1}^\beta f(\Phi^{-1}(x)) |\det(\nabla\Phi^{-1}(x))| \mathrm{d}x.$$

In order for such integrals to be sufficiently preserved when mapping between Ω and $\hat{\Omega}$ we stipulate further that Φ satisfies

$$\int_{\Phi(\Omega)} x_{m+1}^\beta f \mathrm{d}x = \int_\Omega (\Phi(x))_{m+1}^\beta f(\Phi(x)) |\det(\nabla\Phi(x))| \mathrm{d}x \leq C \int_\Omega x_{m+1}^\beta f(\Phi(x)) \mathrm{d}x \tag{2.3.1}$$

and

$$\begin{aligned}\int_{\Phi^{-1}(\hat{\Omega})} x_{m+1}^\beta f \mathrm{d}x &= \int_{\hat{\Omega}} (\Phi^{-1}(x))_{m+1}^\beta f(\Phi^{-1}(x)) |\det(\nabla\Phi^{-1}(x))| \mathrm{d}x \\ &\leq C \int_{\hat{\Omega}} x_{m+1}^\beta f(\Phi^{-1}(x)) \mathrm{d}x \end{aligned} \tag{2.3.2}$$

for a positive constant C. Any bi-Lipschitz map that satisfies (2.3.1) and (2.3.2) will be called a $\mathrm{d}\mu_\beta$*-equivalence from* Ω *to* $\hat{\Omega}$ if $C = C(m,\beta)$ and a *uniform*

$\mathrm{d}\mu_\beta$*-equivalence from* Ω *to* $\hat{\Omega}$ if $C = C(m)$.

A necessary and sufficient condition for Φ to satisfy (2.3.1) and (2.3.2) is that it must preserve the weight function x_{m+1}^β in the following sense. We introduce some further terminology to make this notion precise; if $\Phi : \Omega \to \hat{\Omega}$ satisfies

$$(\Phi(x))_{m+1}^\beta \leq C x_{m+1}^\beta \text{ for } x \in \Omega \tag{2.3.3}$$

and

$$(\Phi^{-1}(x))_{m+1}^\beta \leq C x_{m+1}^\beta \text{ for } x \in \hat{\Omega} \tag{2.3.4}$$

for a positive $C = C(m, \beta)$ then we say Φ is an x_{m+1}^β*-equivalence from* Ω *to* $\hat{\Omega}$ and if $C = C(m)$ then we say Φ is a *uniform* x_{m+1}^β*-equivalence from* Ω *to* $\hat{\Omega}$.

We discuss in detail the bi-Lipschitz, piecewise C^1 with piecewise C^1 inverse maps which allow us to transform between cubes and balls and half-cubes and half-balls. The purpose of our calculations is to show that such a map is a uniform $\mathrm{d}\mu_\beta$ and x_{m+1}^β-equivalence between half-balls and half-cubes with the same centres. All other bi-Lipschitz maps we use will be discussed at their time of use and the required calculations are similar or less tedious than those given below.

First, let us introduce some notation for the sets we consider. Let $y \in \mathbb{R}^{m+1}$, $m \in \mathbb{N}$ and $r > 0$. Typically we will work in $m+1$ dimensional Euclidean space. Consider the $m+1$-dimensional open cubes

$$Q_r^{m+1}(y) = \{x \in \mathbb{R}^{m+1} : x_i \in (y_i - r, y_i + r) \text{ for } i = 1, \dots, m+1\}$$

and the $m+1$-dimensional open balls

$$B_r^{m+1}(y) = \{x \in \mathbb{R}^{m+1} : |x - y| < r\}.$$

For $y \in \partial\mathbb{R}_+^{m+1}$ and $r > 0$ we define the $m+1$-dimensional half-cubes and half-balls

$$Q_r^{+,m+1}(y) = Q_r^{m+1}(y) \cap \mathbb{R}_+^{m+1}$$

and

$$B_r^{+,m+1}(y) = B_r^{m+1}(y) \cap \mathbb{R}_+^{m+1}$$

respectively. In contexts where we do not need to distinguish the dimension, we supress the superscript and use the notation $Q_r^{m+1}(y) = Q_r(y)$, $B_r^{m+1}(y) = B_r(y)$, $Q_r^+(y) = Q_r^{+,m+1}(y)$ and $B_r^+(y) = B_r^{+,m+1}(y)$.

Henceforth we work in $m+1$ dimensions for $m \in \mathbb{N}$. Let $y \in \mathbb{R}^{m+1}$ and consider the cones

$$C_{y,k} = \{x \in \mathbb{R}^{m+1} \backslash \{y\} : |x_k - y_k| = \max_{j \in \{1,\dots,m+1\}} |z_j - y_j|\}$$

and define

$$\Phi_{y,k} : C_{y,k} \to C_{y,k} : x \mapsto |x_k - y_k| \frac{(x-y)}{|x-y|} + y.$$

Then each $\Phi_{y,k}$ is Lipschitz continuous in $C_{y,k}$ and C^1 in the interior of $C_{y,k}$. Furthermore, each $\Phi_{y,k}$ has inverse

$$\Phi_{y,k}^{-1} : C_{y,k} \to C_{y,k} : x \mapsto |x - y| \frac{(x-y)}{|x_k - y_k|} + y,$$

which is Lipschitz continuous in $C_{y,k}$ and C^1 in the interior of $C_{y,k}$. Now we may define a bi-Lipschitz piecewise C^1 map $\Phi_y : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ with piecewise C^1 inverse; let $\Phi_y|_{C_{y,k}} = \Phi_{y,k}$ for $k = 1, \dots, m+1$, let $\Phi_y|_{C_{y,k}} = \Phi_{y,k}$ for one such k whenever the $C_{y,k}$ overlap and define $\Phi_y(y) = y$. Then Φ_y is well defined and bi-Lipschitz but fails to be C^1 on the intersections of the $C_{y,k}$ and at y.

Next we show how transformations between $Q_r^+(y)$ and $B_r^+(y)$, where $y \in \partial\mathbb{R}_+^{m+1}$, under Φ_y and Φ_y^{-1}, affect integrals over these regions. Note that the definition of Φ_y implies $\Phi_y(Q_r^+(y)) = B_r^+(y)$. Recall the equivalence of the Euclidean and maximum distance functions on $\mathbb{R}^{m+1}$, in particular note that

$$\max_{j \in \{1,\dots,m+1\}} |x_j - y_j| \leq |x - y| \leq (m+1)^{\frac{1}{2}} \max_{j \in \{1,\dots,m+1\}} |x_j - y_j|.$$

Let $x \in \mathbb{R}_+^{m+1}$ and suppose further that $x \in C_{y,k}^l$ for some $k \in \{1, \dots, m+1\}$ so that $\max_{j \in \{1,\dots,m+1\}} |x_j - y_j| = |x_k - y_k|$. Then

$$\frac{1}{(m+1)^{\frac{1}{2}}} x_{m+1} \leq (\Phi_y(x))_{m+1} = |x_k - y_k| \frac{x_{m+1}}{|x-y|} \leq x_{m+1} \tag{2.3.5}$$

and

$$x_{m+1} \leq \left(\Phi_y^{-1}(x)\right)_{m+1} = |x - y| \frac{x_{m+1}}{|x_k - y_k|} \leq (m+1)^{\frac{1}{2}} x_{m+1}. \tag{2.3.6}$$

We also calculate

$$\begin{aligned}(\nabla\Phi_{y,k}(x))_{ij} = \operatorname{sgn}(x_k - y_k)\delta_{jk} \frac{x_i - y_i}{|x-y|} + |x_k - y_k| \frac{\delta_{ij}}{|x-y|} \\ - |x_k - y_k| \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^3},\end{aligned}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. We obtain a similar expression for $(\nabla\Phi_y^{-1})_{ij}$ and hence deduce there is a positive constant $c = c(m)$ such that

$$|\nabla\Phi_y(x)|, |\det(\nabla\Phi_y(x))|, |\nabla\Phi_y^{-1}(x)| \text{ and } |\det(\nabla\Phi_y^{-1}(x))| \leq c. \tag{2.3.7}$$

Suppose $f : Q_r^+(y) \to [0, \infty]$ and $g : B_r^+(y) \to [0, \infty]$ are $\mathrm{d}\mu_\beta$ integrable. We combine (2.3.5), (2.3.6) and (2.3.7) with (2.3.1) and (2.3.2) to deduce that

$$\int_{Q_r^+(y)} x_{m+1}^\beta f \mathrm{d}x \leq cc_1^\beta \int_{B_r^+(y)} x_{m+1}^\beta f(\Phi_y^{-1}(x)) \mathrm{d}x \tag{2.3.8}$$

and

$$\int_{B_r^+(y)} x_{m+1}^\beta f \mathrm{d}x \leq cc_2^\beta \int_{Q_r^+(y)} x_{m+1}^\beta f(\Phi_y(x)) \mathrm{d}x, \tag{2.3.9}$$

for two constants $c_1 = c_1(m)$ and $c_2 = c_2(m)$ which arise from (2.3.5) and (2.3.6) respectively. Since $\beta \in (-1, 1)$, we can choose another constant $C = C(m)$ such that $cc_1^\beta \leq C$ and $cc_2^\beta \leq C$. Hence the map Φ_y is both a uniform $\mathrm{d}\mu_\beta$-equivalence and a uniform x_{m+1}^β-equivalence from $Q_r^+(y)$ to $B_r^+(y)$.

2.3.2 Traces of Sobolev Functions

The boundary values of Sobolev functions defined on open $\Omega \subset \mathbb{R}_+^{m+1}$ will play an important role in the variational problems we will consider.

Henceforth we will assume that we are working on domains $\Omega \subset \mathbb{R}_+^{m+1}$ such that a continuous linear trace operator $T : W^{1,p}(\Omega; \mathbb{R}^n) \to L^p(\partial\Omega; \mathbb{R}^n)$ exists for some $p \in (1, 2]$ with $Tv = v|_{\partial\Omega}$ whenever additionally $v \in C(\overline{\Omega}; \mathbb{R}^n)$. This is the case, for instance, on open, bounded $\Omega \subset \mathbb{R}_+^{m+1}$ with Lipschitz boundary, as illustrated in [18] Section 4.3 Theorem 1. In view of Lemmata 2.2.1.3 and 2.2.1.4, we therefore have a continuous linear trace operator $T : W_\beta^{1,2}(\Omega; \mathbb{R}^n) \to L^p(\partial\Omega; \mathbb{R}^n)$ with $Tv = v|_{\partial\Omega}$ whenever additionally $v \in C(\overline{\Omega}; \mathbb{R}^n)$. Here we may choose $p \in (1, 2]$ if $\overline{\Omega} \subset \mathbb{R}_+^{m+1}$, $p \in (1, 2]$ if $\partial\Omega \cap \partial\mathbb{R}_+^{m+1} \neq \emptyset$ and $\beta \in (-1, 0]$ and $p \in (1, \frac{2}{1+\beta})$ if $\partial\Omega \cap \partial\mathbb{R}_+^{m+1} \neq \emptyset$ and $\beta \in (0, 1)$.

The values of functions in $\dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; \mathbb{R}^n)$ on open $\mathcal{O} \subset \mathbb{R}^m \times \{0\}$ will be of particular importance. We will assume that $\mathcal{O} \subset \partial\Omega$ for an Ω such that the trace operators mentioned previously exist, this way we have an interpretation for traces of elements of the homogeneous Sobolev spaces, given by composing the trace operators described previously with the embedding given in Lemma 2.2.1.1.

2.3.3 Poincaré Inequality for $W^{1,2}_\beta$

Poincaré inequalities are a fundamental aspect of both the theory of Sobolev spaces and the analysis of partial differential equations. We will require inequalities regarding the L^2_β distance between a function and its average in terms of its energy.

Before proceeding any further with a discussion of the relevant Poincaré inequalities, we observe a related property of the minimisation of the integral $\int_\Omega |v-\lambda|^2 \mathrm{d}\mu_\beta$ over all constant vectors $\lambda \in \mathbb{R}^n$. Let $\overline{v}_{\Omega,\beta} = \frac{1}{\int_\Omega \mathrm{d}\mu_\beta}\int_\Omega v \mathrm{d}\mu_\beta$ and $\overline{v}_{\Omega,0} = \overline{v}_\Omega = \frac{1}{\int_\Omega \mathrm{d}x}\int_\Omega v\mathrm{d}x$ where $\Omega \subset \mathbb{R}^{m+1}$ is open and bounded and v is integrable on Ω.

Lemma 2.3.3.1. *Suppose $\Omega \subset \mathbb{R}^{m+1}$ is open and bounded and let $v \in W^{1,2}_\beta(\Omega;\mathbb{R}^n)$. Then*

$$\inf_{\lambda\in\mathbb{R}^n}\int_\Omega |v-\lambda|^2\mathrm{d}\mu_\beta = \int_\Omega |v-\overline{v}_{\Omega,\beta}|^2\mathrm{d}\mu_\beta.$$

Proof. To prove the Lemma we calculate the first and second order partial derivatives of the function

$$f:\mathbb{R}^n \to \mathbb{R} : \lambda \to f(\lambda) = \int_\Omega |v-\lambda|^2\mathrm{d}\mu_\beta$$

with respect to the components λ_i, for $i = 1,\ldots,n$, of λ and use calculus to find and determine the nature of any critical points of f.

If λ is a critical point f then $\nabla f(\lambda) = 0$. Hence,

$$\frac{\partial f}{\partial \lambda_i}(\lambda) = 2\int_\Omega (\lambda_i - v_i)\mathrm{d}\mu_\beta = 0$$

for $i = 1,\ldots,n$. Thus $\lambda_i = \overline{(v_i)}_{\Omega,\beta}$ and hence $\lambda = \overline{v}_{\Omega,\beta}$. To see that $\overline{v}_{\Omega,\beta}$ minimises f we calculate the second derivatives of f. We have $\frac{\partial^2 f}{\partial\lambda_i\lambda_j} = 0$ for $i \neq j$ and $\frac{\partial^2 f}{\partial \lambda_i^2} = 2\int_\Omega \mathrm{d}\mu_\beta$. Hence the determinant of the Hessian of f is $2^n\left(\int_\Omega \mathrm{d}\mu_\beta\right)^n > 0$ for every λ in $\mathbb{R}^n$. This implies that f has a local minimum at $\lambda = \overline{v}_{\Omega,\beta}$ and, furthermore, that f is convex. Thus f attains a global minimum at $\lambda = \overline{v}_{\Omega,\beta}$. □

This property allows us to readily deduce a Poincaré inequality for $W^{1,2}_\beta$ functions on any $B_r(y)$ or $Q_r(y)$ with closure contained in $\mathbb{R}^{m+1}_+$.

Lemma 2.3.3.2. *Let $\beta \in (-1,1)$, $y \in \mathbb{R}^{m+1}_+$, $r > 0$ and suppose $B_r(y)$ and $Q_r(y)$ have closures contained in $\mathbb{R}^{m+1}_+$. Let Ω denote either $B_r(y)$ or $Q_r(y)$. Then*

$$\int_\Omega |v-\overline{v}_{\Omega,\beta}|^2\mathrm{d}\mu_\beta \leq Cr^2\frac{\sup_\Omega(x^\beta_{m+1})}{\inf_\Omega(x^\beta_{m+1})}\int_\Omega |\nabla v|^2\mathrm{d}\mu_\beta$$

for a positive constant $C = C(m)$.

Proof. We combine the fact that the weights x_{m+1}^{β} are bounded on Ω with Lemma 2.3.3.1 and apply the Poincaré inequality for $W^{1,2}$. First, notice that Lemma 2.3.3.1 gives

$$\int_{\Omega} |v - \overline{v}_{\Omega,\beta}|^2 \mathrm{d}\mu_\beta \le \int_{\Omega} |v - \overline{v}_{\Omega}|^2 \mathrm{d}\mu_\beta.$$

Since the weights x_{m+1}^{β} are bounded on Ω we have

$$\int_{\Omega} |v - \overline{v}_{\Omega}|^2 \mathrm{d}\mu_\beta \le \sup_{\Omega}(x_{m+1}^{\beta}) \int_{\Omega} |v - \overline{v}_{\Omega}|^2 \mathrm{d}x.$$

The Poincaré inequality for $W^{1,2}(\Omega; \mathbb{R}^n)$ yields

$$\int_{\Omega} |v - \overline{v}_{\Omega}|^2 \mathrm{d}x \le Cr^2 \int_{\Omega} |\nabla v|^2 \mathrm{d}x$$

and using the boundedness of the weights again gives

$$\int_{\Omega} |\nabla v|^2 \mathrm{d}x \le \frac{1}{\inf_{\Omega}(x_{m+1}^{\beta})} \int_{\Omega} |\nabla v|^2 \mathrm{d}\mu_\beta.$$

Combining the above yields the statement of the Lemma. □

The factor $\frac{\sup_{\Omega}(x_{m+1}^{\beta})}{\inf_{\Omega}(x_{m+1}^{\beta})}$ in Lemma 2.3.3.2 is inconvenient in general, depending on the domain and the sign of β, it may blow up or degenerate on domains close to or far from $\partial\mathbb{R}_{+}^{m+1}$. However, on a particular class of balls in $\mathbb{R}_{+}^{m+1}$, as discussed subsequently in section 3.4, this term can actually be bounded above by a constant depending only on m. These inequalities, combined with a Poincaré inequality with respect to the measures $\mathrm{d}\mu_\beta$ on open half-cubes and half-balls with centres on $\partial\mathbb{R}_{+}^{m+1}$, will be sufficient for our purposes.

To prove some of our Lemmata regarding inequalities for functions hereafter, we will reduce the proofs to the case of maps defined on the easiest domain to work with. This will involve the use of bi-Lipschitz maps which are $\mathrm{d}\mu_\beta$-equivalences between the domains as discussed in section 2.3.1. Furthermore we will make use of scale and translation invariance to rescale the domain to unit size and centre it at the origin. The idea is to prove the Lemma on such a domain and then apply this version of the Lemma to a suitably rescaled map to deduce the Lemma for other domains. In the next two Lemmata, we give the details of this process, discussing the use of bi-Lipschitz transformations in Lemma 2.3.3.3 and rescaling in Lemma 2.3.5.2.

There is a general Poincaré inequality for A_2 weights in which the constant may depend on the weight. Next we prove a Poincaré inequality for the weights x_{m+1}^{β} such that the constant is uniform in β and thus independent of these weights.

Lemma 2.3.3.3. *Let $v \in W^{1,2}_{\beta}(\Omega; \mathbb{R}^n)$ where Ω is either a half-ball $B_r^+(y)$ or half-cube $Q_r^+(y)$ for $y \in \partial\mathbb{R}^{m+1}_+$ and $r > 0$. Then*

$$\int_{\Omega} |v - \overline{v}_{\Omega,\beta}|^2 \mathrm{d}\mu_\beta \leq Cr^2 \int_{\Omega} |\nabla v|^2 \mathrm{d}\mu_\beta \tag{2.3.10}$$

for a constant C which only depends on m.

Proof. Throughout, C denotes a postive constant that depends only on m. First we reduce the inequality for half-balls to the case of half-cubes. To this end, suppose that (2.3.10) holds for a half-cube $Q_r^+(y)$ and let $\Phi_y : Q_r^+(y) \to B_r^+(y)$ denote the bi-Lipschitz map described in section 2.3.1. An application of Lemma 2.3.3.1 gives

$$\int_{B_r^+(y)} |v - \overline{v}_{B_r^+(y),\beta}|^2 \mathrm{d}\mu_\beta \leq \int_{B_r^+(y)} |v - \overline{(v \circ \Phi_y)}_{Q_r^+(y),\beta}|^2 \mathrm{d}\mu_\beta. \tag{2.3.11}$$

Since Φ_y is a uniform $\mathrm{d}\mu_\beta$-equivalence from $Q_r^+(y)$ to $B_r^+(y)$, as described in section 2.3.1, there is a constant $c = c(m)$ such that

$$\int_{B_r^+(y)} |v - \overline{(v \circ \Phi_y)}_{Q_r^+(y),\beta}|^2 \mathrm{d}\mu_\beta \leq c \int_{Q_r^+(y)} |v \circ \Phi_y(x) - \overline{(v \circ \Phi_y)}_{Q_r^+(y),\beta}|^2 \mathrm{d}\mu_\beta. \tag{2.3.12}$$

The Poincaré inequality for half-cubes, which we have assumed holds for now, gives

$$\int_{Q_r^+(y)} |v \circ \Phi_y(x) - \overline{(v \circ \Phi_y)}_{Q_r^+(y),\beta}|^2 \mathrm{d}\mu_\beta \leq Cr^2 \int_{Q_r^+(y)} |\nabla (v \circ \Phi_y)|^2 \mathrm{d}\mu_\beta, \tag{2.3.13}$$

and since Φ_y has bounded gradient in terms of a constant depending only on m, as in (2.3.7), we have

$$\int_{Q_r^+(y)} |\nabla (v \circ \Phi_y)|^2 \mathrm{d}\mu_\beta \leq C \int_{Q_r^+(y)} |\nabla v(\Phi_y(x))|^2 \mathrm{d}\mu_\beta. \tag{2.3.14}$$

The map Φ_y^{-1} also satisfies (2.3.7), hence

$$\int_{Q_r^+(y)} |\nabla v(\Phi_y(x))|^2 \mathrm{d}\mu_\beta \leq C \int_{B_r^+(y)} |\nabla v|^2 \mathrm{d}\mu_\beta. \tag{2.3.15}$$

Combining (2.3.11), (2.3.12), (2.3.13), (2.3.14) and (2.3.15) yields (2.3.10). Hence we may assume $\Omega = Q_r^+(y)$.

If $\beta = 0$ then (2.3.10) is the Poincaré inequality in $W^{1,2}(Q_r^+(y); \mathbb{R}^n)$. The approach we use to prove the Lemma for other β requires us to distinguish the cases $\beta \in (-1, 0)$ and $\beta \in (0, 1)$; however, the underlying idea in each case is the same. We will take advantage of Lemma 2.3.3.1 and prove that

$$\int_{Q_r^+(y)} x_{m+1}^\beta |v - \lambda|^2 \mathrm{d}x \leq Cr^2 \int_{Q_r^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \tag{2.3.16}$$

for a $\lambda \in \mathbb{R}^n$ depending on β. Combined with Lemma 2.3.3.1 this yields (2.3.10).

Observe that (2.3.16) is invariant under re-scaling and translation in the variables $x_1, \ldots, x_m$ in the sense that if (2.3.16) holds on $Q_1^+(0)$, then we can obtain (2.3.16) on any $Q_r^+(y)$ for $r > 0, y \in \partial\mathbb{R}_+^{m+1}$, with the same constant C, by applying the inequality on $Q_1^+(0)$ to the rescaled function $v_r(x) = v(rx + y)$. Hence we may assume $r = 1$ and $y = 0$.

In order to find $\lambda \in \mathbb{R}^n$ as in (2.3.16) we consider the cases $\beta \in (-1, 0)$ and $\beta \in (0, 1)$ separately. However, in either case we consider

$$\int_{Q_1^+(0)} h(x_{m+1}) |v - \lambda|^2 \mathrm{d}x \tag{2.3.17}$$

for a function $h : (0, 1) \to (0, \infty)$ closely related to the weights x_{m+1}^β. The idea is to re-write (2.3.17), using Fubini's Theorem, in terms of the integral of the derivative of h multiplied by a term to which we may apply the Poincaré inequality for $W^{1,2}$ functions, provided λ is chosen accordingly. We use a second application of Fubini's Theorem to write the integrals in terms of h again which will conclude the proofs.

The case $\beta \in (0, 1)$

We will show that for $\beta \in (0, 1)$, a sufficient choice of $\lambda \in \mathbb{R}^n$ to ensure that v satisfies (2.3.16) on $Q_1^+(0)$ is $\lambda = \overline{v}_\Omega$, where $\Omega = (-1, 1)^m \times (\frac{1}{2}, 1)$. For convenience we denote $(-1, 1)^m$ by Q' so that $Q_1^+(0) = Q' \times (0, 1)$ and $\Omega = Q' \times (\frac{1}{2}, 1)$. Furthermore, $\mathrm{d}x'$ denotes the m-dimensional Lebesgue measure.

Define the function

$$h(t) = \begin{cases} t^\beta & \text{for } t \in [0, \frac{1}{2}] \\ \frac{1}{2^\beta} & \text{for } t \in (\frac{1}{2}, 1]. \end{cases}$$

Note $h'(t) = 0$ for $t \in (\frac{1}{2}, 1]$ and consider (2.3.17) with this choice of h. We have

$$\int_{Q_1^+(0)} h(x_{m+1})|v - \overline{v}_\Omega|^2 \mathrm{d}x = \int_{Q'} \int_0^1 \int_0^{x_{m+1}} h'(s)\mathrm{d}s|v - \overline{v}_\Omega|^2 \mathrm{d}x_{m+1}\mathrm{d}x'. \quad (2.3.18)$$

An application of Fubini's Theorem yields

$$\begin{aligned}\int_{Q'} \int_0^1 \int_0^{x_{m+1}} h'(s)\mathrm{d}s|v - \overline{v}_\Omega|^2 \mathrm{d}x_{m+1}\mathrm{d}x' &= \int_{Q'} \int_0^1 \int_s^1 h'(s)|v - \overline{v}_\Omega|^2 \mathrm{d}x_{m+1}\mathrm{d}s\mathrm{d}x' \\ &= \int_0^{\frac{1}{2}} h'(s) \int_{Q'} \int_s^1 |v - \overline{v}_\Omega|^2 \mathrm{d}x_{m+1}\mathrm{d}x'\mathrm{d}s.\end{aligned} \quad (2.3.19)$$

Now we use the Poincaré inequality for $W^{1,2}$ functions. It follows from [21] Chapter 7 equation 7.45 that for all $s \in (0, \frac{1}{2}]$ we have

$$\begin{aligned}&\int_{Q'} \int_s^1 |v - \overline{v}_\Omega|^2 \mathrm{d}x_{m+1}\mathrm{d}x' \\ &\le \left(\frac{C}{|\Omega|}\right)^{\frac{2m}{m+1}} \mathrm{diam}(Q' \times [s,1])^{2(m+1)} \int_{Q'} \int_s^1 |\nabla u|^2 \mathrm{d}x_{m+1}\mathrm{d}x' \\ &\le C \int_{Q'} \int_s^1 |\nabla u|^2 \mathrm{d}x_{m+1}\mathrm{d}x'.\end{aligned} \quad (2.3.20)$$

Combining (2.3.18), (2.3.19) and (2.3.20) gives

$$\int_{Q_1^+(0)} h(x_{m+1})|v - \overline{v}_\Omega|^2 \mathrm{d}x \le C \int_0^{\frac{1}{2}} h'(s) \int_{Q'} \int_s^1 |\nabla v|^2 \mathrm{d}x_{m+1}\mathrm{d}x'\mathrm{d}s. \quad (2.3.21)$$

Now we change the order of integration again using Fubini's Theorem. We have

$$\begin{aligned}
\int_0^{\frac{1}{2}} h'(s) \int_{Q'} \int_s^1 |\nabla v|^2 \mathrm{d}x_{m+1} \mathrm{d}x' \mathrm{d}s &= \int_{Q'} \int_0^{\frac{1}{2}} h'(s) \int_s^1 |\nabla v|^2 \mathrm{d}x_{m+1} \mathrm{d}s \mathrm{d}x' \\
&= \int_{Q'} \int_0^{\frac{1}{2}} \int_0^{x_{m+1}} h'(s) \mathrm{d}s |\nabla v|^2 \mathrm{d}x_{m+1} \mathrm{d}x' \\
&\quad + \int_{Q'} \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} h'(s) \mathrm{d}s |\nabla v|^2 \mathrm{d}x_{m+1} \mathrm{d}x' \\
&= \int_{Q'} \int_0^{\frac{1}{2}} h(x_{m+1}) |\nabla v|^2 \mathrm{d}x_{m+1} \mathrm{d}x' \\
&\quad + \frac{1}{2^\beta} \int_{Q'} \int_{\frac{1}{2}}^1 |\nabla v|^2 \mathrm{d}x_{m+1} \mathrm{d}x' \\
&= \int_{Q_1^+(0)} h(x_{m+1}) |\nabla v|^2 \mathrm{d}x. \qquad (2.3.22)
\end{aligned}$$

We combine (2.3.22) with (2.3.21) to see that

$$\int_{Q_1^+(0)} h(x_{m+1}) |v - \overline{v}_\Omega|^2 \mathrm{d}x \leq C \int_{Q_1^+(0)} h(x_{m+1}) |\nabla v|^2 \mathrm{d}x. \qquad (2.3.23)$$

This estimate holds independently of $\beta \in (0,1)$. Lastly notice that

$$\int_{Q_1^+(0)} h(x_{m+1}) |f| \mathrm{d}x \leq \int_{Q_1^+(0)} x_{m+1}^\beta |f| \mathrm{d}x \leq C \int_{Q_1^+(0)} h(x_{m+1}) |f| \mathrm{d}x \qquad (2.3.24)$$

for any integrable f and a constant C that can be chosen independently of $\beta \in (0,1)$. Thus, combining (2.3.23) and (2.3.24) shows that (2.3.16) holds with $\lambda = \overline{v}_\Omega$ and hence we apply Lemma 2.3.3.1 to conclude the proof for $\beta \in (0,1)$.

The case $\beta \in (-1,0)$

Let $\beta \in (-1,0)$. Then $v \in W^{1,2}(Q_1^+(0); \mathbb{R}^n)$ by Lemma 2.2.1.3. Thus we may apply the trace operator $T : W^{1,2}(Q_1^+(0); \mathbb{R}^n) \to L^2(Q' \times \{0\}; \mathbb{R}^n)$ to v. We will show that a sufficient choice of $\lambda \in \mathbb{R}^n$ to ensure v satisfies (2.3.16) on $Q_1^+(0)$ is $\lambda = \overline{(Tv)}_{Q'}$.

Define

$$h(t) = t^\beta - 1$$

for $t \in (0,1]$ and consider (2.3.17) again. We have

$$\int_{Q_1^+(0)} h(x_{m+1})|v - \overline{(Tv)}_{Q'}|^2 \mathrm{d}x = -\int_{Q'} \int_0^1 \int_{x_{m+1}}^1 h'(s)\mathrm{d}s|v - \overline{(Tv)}_{Q'}|^2 \mathrm{d}x' \mathrm{d}x_{m+1}. \tag{2.3.25}$$

We apply Fubini's Theorem to see that

$$\begin{aligned}
&\int_{Q'} \int_0^1 \int_{x_{m+1}}^1 h'(s)\mathrm{d}s|v - \overline{(Tv)}_{Q'}|^2 \mathrm{d}x_{m+1}\mathrm{d}x' \\
&= \int_{Q'} \int_0^1 \int_0^s h'(s)|v - \overline{(Tv)}_{Q'}|^2 \mathrm{d}x_{m+1}\mathrm{d}s\mathrm{d}x' \\
&= \int_0^1 h'(s) \int_{Q'} \int_0^s |v - \overline{(Tv)}_{Q'}|^2 \mathrm{d}x_{m+1}\mathrm{d}x'\mathrm{d}s.
\end{aligned} \tag{2.3.26}$$

To conclude, we need to use a variant of the Poincaré inequality for $W^{1,2}$ functions, namely

$$\int_{Q'} \int_0^s |v - \overline{(Tv)}_{Q'}|^2 \mathrm{d}x_{m+1}\mathrm{d}x' \leq C \int_{Q'} \int_0^s |\nabla v|^2 \mathrm{d}x_{m+1}\mathrm{d}x' \tag{2.3.27}$$

where $s \in (0,1]$ and C is a positive constant independent of s. We digress to prove this assertion.

A Poincaré Inequality Involving Traces

First we show that it suffices to prove (2.3.27) for $s = 1$. To this end, suppose the inequality is true for $s = 1$. That is, assume that

$$\int_{Q'} \int_0^1 |v - \overline{(Tv)}_{Q'}|^2 \mathrm{d}x_{m+1}\mathrm{d}x' \leq C \int_{Q'} \int_0^1 |\nabla v|^2 \mathrm{d}x_{m+1}\mathrm{d}x'.$$

Define $v_s(x) = v(x', sx_{m+1})$ for $s \in (0,1]$. Then $Tv = Tv_s$ for every such s. Using the change of variables $x_{m+1} \mapsto sx_{m+1}$ we calculate

$$\begin{aligned}
\int_{Q'} \int_0^s |v - \overline{(Tv)}_{Q'}|^2 \mathrm{d}x_{m+1} \mathrm{d}x' &= s \int_{Q'} \int_0^1 |v_s - \overline{(Tv_s)}_{Q'}|^2 \mathrm{d}x_{m+1} \mathrm{d}x' \\
&\leq sC \int_{Q'} \int_0^1 |\nabla(v_s)|^2 \mathrm{d}x_{m+1} \mathrm{d}x' \\
&\leq sC \int_{Q'} \int_0^1 |\nabla v(x', sx_{m+1})|^2 \mathrm{d}x_{m+1} \mathrm{d}x' \\
&= C \int_{Q'} \int_0^s |\nabla v|^2 \mathrm{d}x_{m+1} \mathrm{d}x'.
\end{aligned}$$

It follows that

$$\int_{Q'} \int_0^s |v - \overline{(Tv)}_{Q'}|^2 \mathrm{d}x_{m+1} \mathrm{d}x' \leq C \int_{Q'} \int_0^s |\nabla v|^2 \mathrm{d}x_{m+1} \mathrm{d}x'$$

for $s \in (0,1]$ where C is a positive constant independent of s. Hence to show (2.3.27) holds independently of $s \in (0,1]$ we only need prove the inequality for $s = 1$.

To proceed, we adapt one of the many proofs of the Poincaré inequality for $W^{1,2}$ functions, see for example, [46] section 1.3, Lemma 2. Suppose, for a contradiction, that for every $C = C(m)$ there is a $v \in W^{1,2}(Q_1^+(0), \mathbb{R}^n)$ such that

$$C \int_{Q_1^+(0)} |\nabla v|^2 \mathrm{d}x_{m+1} \mathrm{d}x' < \int_{Q'} \int_0^1 |v - \overline{(Tv)}_{Q'}|^2 \mathrm{d}x.$$

In particular, we choose a sequence $(v_k)_{k \in \mathbb{N}}$ of $v_k \in W^{1,2}(Q_1^+(0), \mathbb{R}^n)$ satisfying

$$\int_{Q_1^+(0)} |\nabla v_k|^2 \mathrm{d}x < \frac{1}{k} \int_{Q_1^+(0)} |v_k - \overline{(Tv_k)}_{Q'}|^2 \mathrm{d}x. \tag{2.3.28}$$

Define,

$$w_k(x) = \frac{v_k(x) - \overline{(Tv_k)}_{Q'}}{\left(\int_{Q_1^+(0)} |v_k - \overline{(Tv_k)}_{Q'}|^2 \mathrm{d}x \right)^{\frac{1}{2}}}.$$

Then $\int_{Q_1^+(0)} |w_k|^2 \mathrm{d}x = 1$, $\int_{Q'} Tw_k \mathrm{d}x' = 0$ and

$$\nabla w_k = \frac{\nabla v_k}{\left(\int_{Q_1^+(0)} |v - \overline{(Tv_k)}_{Q'}|^2 \mathrm{d}x \right)^{\frac{1}{2}}}$$

for all k. We substitute this expression for ∇w_k into (2.3.28) to see that

$$\int_{Q_1^+(0)} |\nabla w_k|^2 \mathrm{d}x < \frac{1}{k}. \tag{2.3.29}$$

Thus w_k is a bounded sequence in $W^{1,2}(Q_1^+(0);\mathbb{R}^n)$ and so the Rellich-Kondrachov compactness Theorem yields a subsequence $(w_{k_j})_{j\in\mathbb{N}}$ which converges strongly in $L^2(Q_1^+(0);\mathbb{R}^n)$ to a limit $w \in W^{1,2}(Q_1^+(0);\mathbb{R}^n)$. Extracting a subsequence again, if necessary, we may assume that $w_{k_j} \to w$ almost everywhere as $j \to \infty$. Furthermore, $\nabla w_{k_j} \to \nabla w$ weakly in L^2 since the unit ball in a Hilbert spaces is weakly compact. Hence, in view of (2.3.29) and the lower semicontinuity of the L^2 norm it follows that $\nabla w = 0$ almost everywhere in $Q_1^+(0)$. Therefore w is constant. Since $w_{k_j} \to w$ in L^2 it follows that $w \neq 0$ because $\int_{Q_1^+(0)} |w|^2 \mathrm{d}x = \lim_{j\to\infty} \int_{Q_1^+(0)} |w_{k_j}|^2 \mathrm{d}x = 1$. However, (2.3.29) implies $\int_{Q_1^+(0)} |\nabla w_k|^2 \mathrm{d}x \to 0 = \int_{Q_1^+(0)} |\nabla w|^2 \mathrm{d}x$ and since ∇w_k to ∇w weakly in L^2, we deduce ∇w_k converges to ∇w strongly in L^2. Moreover, $\int_{Q'} T w_{k_j} \mathrm{d}x' = 0$ for all j and the continuity of the trace operator thus implies $\int_{Q'} T w \mathrm{d}x' = 0$. As w is constant, its trace is constant and we must have $Tw = 0$. The only constant function with zero trace is the zero function but we have already shown that w is non-zero which is a contradiction. Thus (2.3.27) holds.

Return to the proof of the Poincaré Inequality for $\beta \in (-1, 0)$.

Recall (2.3.26). In view of (2.3.27), noting that $h'(t) \leq 0$, we see that

$$\begin{aligned} &-\int_0^1 h'(s) \int_{Q'} \int_0^s |v - \overline{(Tv)}_{Q'}|^2 \mathrm{d}x_{m+1} \mathrm{d}x' \mathrm{d}s \\ &\leq -C \int_0^1 h'(s) \int_{Q'} \int_0^s |\nabla v|^2 \mathrm{d}x_{m+1} \mathrm{d}x' \mathrm{d}s. \end{aligned} \tag{2.3.30}$$

Hence, combining (2.3.25), (2.3.26) and (2.3.30) and applying Fubini's Theorem once more we have

$$\begin{aligned} \int_{Q_1^+(0)} h(x_{m+1}) |v - \overline{(Tv)}_{Q'}|^2 \mathrm{d}x &\leq -C \int_0^1 h'(s) \int_{Q'} \int_0^s |\nabla v|^2 \mathrm{d}x_{m+1} \mathrm{d}x' \mathrm{d}s \\ &= C \int_{Q_1^+(0)} h(x_{m+1}) |\nabla v|^2 \mathrm{d}x. \end{aligned}$$

We rearrange the above, noting that $1 \leq x_{m+1}^{\beta}$ on $Q_1^+(0)$, and apply (2.3.27) with $s = 1$ to see that

$$\begin{aligned}\int_{Q_1^+(0)} x_{m+1}^{\beta}|v - \overline{(Tv)}_{Q'}|^2 \mathrm{d}x &\leq \int_{Q_1^+(0)} |v - \overline{(Tv)}_{Q'}|^2 \mathrm{d}x \\ &\quad + C\int_{Q_1^+(0)} h(x_{m+1})|\nabla v|^2 \mathrm{d}x \\ &\leq C\int_{Q_1^+(0)} x_{m+1}^{\beta}|\nabla v|^2 \mathrm{d}x.\end{aligned}$$

Hence (2.3.16) holds with $\lambda = \overline{Tv}_{Q'}$ and we apply Lemma 2.3.3.1 to conclude the proof for $\beta \in (-1, 0)$. $\square$

It will be useful to have a version of the Poincaré inequality on $B_r^+(y)$, with $y_{m+1} = 0$ but with $\overline{v}_{B_r^+(y),\beta}$ replaced by $\overline{v}_{B_{\theta r}^+(y),\beta}$ for $\theta \in (0,1)$. Such a statement is proved using the Poincaré inequality, Lemma 2.3.3.3.

Lemma 2.3.3.4. *Let $v \in W_{\beta}^{1,2}(\Omega_r; \mathbb{R}^n)$ where Ω_r is either a half-ball $B_r^+(y)$ or half-cube $Q_r^+(y)$ with $y \in \partial\mathbb{R}_+^{m+1}$ and $r > 0$. Let $\theta \in (0,1)$. Then*

$$\int_{\Omega_r} |v - \overline{v}_{\Omega_{\theta r},\beta}|^2 \mathrm{d}\mu_{\beta} \leq C\theta^{-(1+m+\beta)} r^2 \int_{\Omega_r} |\nabla v|^2 \mathrm{d}\mu_{\beta} \tag{2.3.31}$$

for a constant $C = C(m)$.

Proof. We have

$$\begin{aligned}\int_{\Omega_r} |v - \overline{v}_{\Omega_{\theta r},\beta}|^2 \mathrm{d}\mu_{\beta} &\leq 2\int_{\Omega_r} |\overline{v}_{\Omega_r,\beta} - \overline{v}_{\Omega_{\theta r},\beta}|^2 \mathrm{d}\mu_{\beta} + 2\int_{\Omega_r} |v - \overline{v}_{\Omega_r,\beta}|^2 \mathrm{d}\mu_{\beta} \\ &= 2r^{1+m+\beta}\int_{\Omega_1} \mathrm{d}\mu_{\beta} |\overline{v}_{\Omega_r,\beta} - \overline{v}_{\Omega_{\theta r},\beta}|^2 + 2\int_{\Omega_r} |v - \overline{v}_{\Omega_r,\beta}|^2 \mathrm{d}\mu_{\beta}.\end{aligned} \tag{2.3.32}$$

The Poincaré inequality, Lemma 2.3.3.3, yields

$$\int_{\Omega_r} |v - \overline{v}_{\Omega_r,\beta}|^2 \mathrm{d}\mu_{\beta} \leq Cr^2 \int_{\Omega_r} |\nabla v|^2 \mathrm{d}\mu_{\beta} \tag{2.3.33}$$

so to conclude, we must consider the remaining term in (2.3.32). An application of Hölder's inequality gives

$$
\begin{aligned}
|\overline{v}_{\Omega_r,\beta} - \overline{v}_{\Omega_{\theta r},\beta}|^2 &\leq \left(\int_{\Omega_{\theta r}} \mathrm{d}\mu_\beta\right)^{-1} \int_{\Omega_{\theta r}} |v - \overline{v}_{\Omega_r,\beta}|^2 \mathrm{d}\mu_\beta \\
&\leq (\theta r)^{-(1+m+\beta)} \left(\int_{\Omega_1} \mathrm{d}\mu_\beta\right)^{-1} \int_{\Omega_r} |v - \overline{v}_{\Omega_r,\beta}|^2 \mathrm{d}\mu_\beta.
\end{aligned}
$$

Thus, using Lemma 2.3.3.3 again, we find

$$
|\overline{v}_{\Omega_r,\beta} - \overline{v}_{\Omega_{\theta r},\beta}|^2 \leq (\theta r)^{-(1+m+\beta)} \left(\int_{\Omega_1} \mathrm{d}\mu_\beta\right)^{-1} C r^2 \int_{\Omega_r} |\nabla v|^2 \mathrm{d}\mu_\beta. \tag{2.3.34}
$$

Combining (2.3.32), (2.3.33) and (2.3.34) concludes the proof. □

2.3.4 Pointwise Bounds for Functions in Terms of Their Integrals on an Interval

We discuss bounding a measurable function, defined on an interval, in terms of the integral of the function over the interval. Let $(a,b) \subset \mathbb{R}$ with $a < b$ and consider the measure $\mathrm{d}\mu_\beta^1 = |x|^\beta \mathrm{d}x$ on (a,b). Suppose $f : (a,b) \to [0,\infty]$ is a $\mathrm{d}\mu_\beta^1$-integrable function and let $\theta \in (0,1)$. Then for all $x \in (a,b)$, with the exception of a set of $\mathrm{d}\mu_\beta^1$-measure $\theta \int_a^b \mathrm{d}\mu_\beta^1$, we have

$$
f(x) \leq \frac{1}{\theta \int_a^b \mathrm{d}\mu_\beta^1} \int_a^b f \mathrm{d}\mu_\beta^1. \tag{2.3.35}
$$

Otherwise the reverse inequality would hold on a set P of measure greater than $\theta \int_a^b \mathrm{d}\mu_\beta^1$ and integrating over P would yield

$$
\int_a^b f \mathrm{d}\mu_\beta^1 < \frac{\int_P \mathrm{d}\mu_\beta^1}{\theta \int_a^b \mathrm{d}\mu_\beta^1} \int_a^b f \mathrm{d}\mu_\beta^1 < \int_P f \mathrm{d}\mu_\beta^1
$$

which is a contradiction.

Similary, we deduce an analogous weaker statement. There always exists an $x \in (a,b)$ such that

$$
f(x) \leq \frac{1}{\int_a^b \mathrm{d}\mu_\beta^1} \int_a^b f \mathrm{d}\mu_\beta^1 \tag{2.3.36}
$$

otherwise the reverse inequality would hold for every $x \in (a,b)$ and integrating over (a, b) would give the contradiction

$$\int_a^b f \mathrm{d}\mu_\beta^1 < \int_a^b f \mathrm{d}\mu_\beta^1.$$

2.3.5 Compactness of the Embedding $W_\beta^{1,2} \hookrightarrow L_\beta^2$

Compact embeddings provide a means to analyse bounded sequences of solutions to partial differential equations. We will, for instance, need to consider sequences of solutions to (2.0.1).

Compactness of the Embedding on Domains with Closure in $\mathbb{R}_+^{m+1}$

When $\Omega \subset \mathbb{R}_+^{m+1}$ is bounded, open and satisfies $\overline{\Omega} \subset \mathbb{R}_+^{m+1}$ we may directly take advantage of the Rellich Kondrachov Compactness Theorem to deduce that the embedding $W_\beta^{1,2}(\Omega; \mathbb{R}^n) \hookrightarrow L_\beta^2(\Omega; \mathbb{R}^n)$ is compact. We summarise this fact in a Lemma, which follows directly from [46] Section 1.3, Lemma 1, along with some additional properties of bounded sequences in $W_\beta^{1,2}(\Omega; \mathbb{R}^n)$.

Lemma 2.3.5.1. *Let $\beta \in (-1, 1)$ and suppose $\Omega \subset \mathbb{R}_+^{m+1}$ is bounded and open with $\overline{\Omega} \subset \mathbb{R}_+^{m+1}$. Suppose that $(v_j)_{j \in \mathbb{N}}$ is a sequence in $W_\beta^{1,2}(\Omega; \mathbb{R}^n)$ which satisfies $\sup_j ||v_j||_{W_\beta^{1,2}(\Omega;\mathbb{R}^n)} < \infty$. Then there exists a subsequence $(v_{j_k})_{k \in \mathbb{N}}$ and a $v \in W_\beta^{1,2}(\Omega; \mathbb{R}^n)$ such that*

1. $v_{j_k} \rightharpoonup v$ *in* $W_\beta^{1,2}(\Omega; \mathbb{R}^n)$
2. $v_{j_k} \to v$ *in* $L_\beta^2(\Omega; \mathbb{R}^n)$
3. $\int_\Omega x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega x_{m+1}^\beta |\nabla v_{j_k}|^2 \mathrm{d}x.$

Proof. The statement of the Lemma for $\beta = 0$ is Lemma 1 of Section 1.3 in [46]. For $\beta \neq 0$, statement 1 is a consequence of the weak compactness of the unit ball in the Hilbert spaces $W_\beta^{1,2}(\Omega; \mathbb{R}^n)$. Recall that $W_\beta^{1,2}(\Omega; \mathbb{R}^n) = W^{1,2}(\Omega; \mathbb{R}^n)$ by Lemma 2.2.1.2 and similarly $L_\beta^2(\Omega; \mathbb{R}^n) = L^2(\Omega; \mathbb{R}^n)$ on the Ω in consideration. Since the two norms $||\cdot||_{W_\beta^{1,2}(\Omega;\mathbb{R}^n)}$ and $||\cdot||_{W^{1,2}(\Omega;\mathbb{R}^n)}$ and the two norms $||\cdot||_{L_\beta^2(\Omega;\mathbb{R}^n)}$ and $||\cdot||_{L^2(\Omega;\mathbb{R}^n)}$ are equivalent, we conclude that statement 2 for sequences in $W_\beta^{1,2}(\Omega; \mathbb{R}^n)$ follows from the case for $W^{1,2}(\Omega; \mathbb{R}^n)$. Statement 2 combined with the lower semi-continuity of the Hilbert space norm $||\cdot||_{W_\beta^{1,2}(\Omega;\mathbb{R}^n)}$ yields statement 3. $\square$

Compactness of the Embedding on Half-Cubes and Half-Balls

We require a counterpart to Lemma 2.3.5.1 concerning the compactness of the embedding $W^{1,2}_\beta(\Omega;\mathbb{R}^n) \hookrightarrow L^2_\beta(\Omega;\mathbb{R}^n)$ when Ω is a half-cube or half-ball.

Lemma 2.3.5.2. *Let $r > 0$, $y \in \partial\mathbb{R}^{m+1}_+$ and suppose $(v_j)_{j\in\mathbb{N}}$ is a sequence in $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ with $\sup_j ||v_j||_{W^{1,2}_\beta(\Omega;\mathbb{R}^n)} < \infty$ where Ω is either $Q^+_r(y)$ or $B^+_r(y)$. Then there exists a subsequence $(v_{j_k})_{k\in\mathbb{N}}$ and a $v \in W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ such that*

1. *$v_{j_k} \rightharpoonup v$ in $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$*
2. *$v_{j_k} \to v$ in $L^2_\beta(\Omega;\mathbb{R}^n)$*
3. *$\int_\Omega x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega x^\beta_{m+1}|\nabla v_{j_k}|^2 \mathrm{d}x$.*

Proof. For $\beta = 0$ the proof is the same as for Lemma 2.3.5.1, a statement and proof of which can be found in [46] section 1.3 Lemma 1. Henceforth we suppose $\beta \in (-1,1)\backslash\{0\}$.

Statement 1 follows from the weak sequential compactness of the unit ball in a Hilbert space. Furthermore, statement 3 follows from statement 2 and the lower semi-continuity of a Hilbert space norm. Hence, the main task is to prove statement 2.

We may assume that Ω is a half-cube which we justify as follows. If the Lemma is true on a $Q^+_r(y)$ then statement 1 for sequences of maps with domain $B^+_r(y)$ follows in the same way as for the domains $Q^+_r(y)$. For any sequence $(v_k)_{k\in\mathbb{N}}$ with $v_k \in W^{1,2}_\beta(B^+_r(y);\mathbb{R}^n)$, we can apply the Lemma to the sequence $(v_k \circ \Phi_y)_{k\in\mathbb{N}}$, where $\Phi_y : Q^+_r(y) \to B^+_r(y)$ is the bi-Lipschitz map defined in section 2.3.1, to deduce statement 2 for $(v_k)_{k\in\mathbb{N}}$. Statement 3 then also follows as described for the domains $Q^+_r(y)$.

Now we show that statement 2 is invariant under rescaling and translations with respect to x_i for $i = 1,\dots,m$. Suppose that the statement is true for any sequence $(v_j)_{j\in\mathbb{N}}$ of $v_j \in W^{1,2}_\beta(Q^+_1(0);\mathbb{R}^n)$ satisfying the assumptions of the Lemma. Then we can obtain the statement for any sequence $(\hat{v}_j)_{j\in\mathbb{N}}$ of $\hat{v}_j \in W^{1,2}_\beta(Q^+_r(y);\mathbb{R}^n)$ which satisfies the assumptions of the Lemma, by applying statement 2 on $Q^+_1(0)$ to the sequence $v_j(x) = \hat{v}_j(rx+y)$ defined for $x \in Q^+_1(0)$. A change of variables shows that

$$||v_j||^2_{L^2_\beta(Q^+_1(0);\mathbb{R}^n)} = r^{-(1+m+\beta)}||\hat{v}_j||^2_{L^2_\beta(Q^+_r(y);\mathbb{R}^n)} \tag{2.3.37}$$

and

$$||\nabla v_j||_{L^2_\beta(Q^+_1(0);\mathbb{R}^n)} = r^{1-m-\beta}||\nabla \hat{v}_j||_{L^2_\beta(Q^+_r(y);\mathbb{R}^n)}. \tag{2.3.38}$$

Hence the sequence $(v_j)_{j\in\mathbb{N}}$ satisfies the assumptions of the Lemma on $Q_1^+(0)$ and there is a $v \in W_\beta^{1,2}(Q_1^+(0);\mathbb{R}^n)$ such that a subsequence $(v_{j_k})_{k\in\mathbb{R}^n}$ converges to v in $L_\beta^2(Q_1^+(0);\mathbb{R}^n)$. We define $\hat{v}(x) = v(\frac{x-y}{r})$ for $x \in Q_r^+(y)$. Both (2.3.37) and (2.3.38) hold with v_j replaced by v and $\hat{v}_j$ replaced by $\hat{v}$; we conclude that $\hat{v} \in W_\beta^{1,2}(Q_r^+(y);\mathbb{R}^n)$ and $(\hat{v}_{j_k})_{k\in\mathbb{R}^n}$ converges to $\hat{v}$ in $L_\beta^2(Q_r^+(y);\mathbb{R}^n)$. Hence we only need show statement 2 holds on $Q_1^+(0)$.

Suppose $(v_j)_{j\in\mathbb{N}}$ is a sequence with $v_j \in W_\beta^{1,2}(Q_1^+(0);\mathbb{R}^n)$ for every j, which satisfies

$$\sup_{j\in\mathbb{N}} ||v_j||_{W_\beta^{1,2}(Q_1^+(0);\mathbb{R}^n)} \leq M \tag{2.3.39}$$

for some positive constant M. Without relabelling the index, suppose $(v_j)_{j\in\mathbb{N}}$ is also the subsequence which satisfies $v_j \rightharpoonup v$ for $v \in W_\beta^{1,2}(Q_1^+(0);\mathbb{R}^n)$. Since $W_\beta^{1,2}(Q_1^+(0);\mathbb{R}^n)$ is a Hilbert space, the norm on this space is weakly lower semi-continuous and hence we have

$$||v||_{W_\beta^{1,2}(Q_1^+(0);\mathbb{R}^n)} \leq M. \tag{2.3.40}$$

Let $Q' = (-1,1)^m$ so that $Q_1^+(0) = Q' \times (0,1)$ and define

$$Q_i = \left\{(x', x_{m+1}) \in Q_1^+(0) : \frac{1}{i+1} < x_{m+1} \leq 1\right\}$$

for $i \in \mathbb{N}$. We will take advantage of the compactness of the embeddings $W_\beta^{1,2}(Q_i;\mathbb{R}^n) \hookrightarrow L_\beta^2(Q_i;\mathbb{R}^n)$ provided by Lemma 2.3.5.1 in order to construct a subsequence of $(v_j)_{j\in\mathbb{N}}$ which converges to v in $L_\beta^2(Q_1^+(0);\mathbb{R}^n)$.

Notice that in view of (2.3.39), for each $i \in \mathbb{N}$ we have

$$\sup_{j\in\mathbb{N}} ||v_j||_{W_\beta^{1,2}(Q_i;\mathbb{R}^n)} \leq M. \tag{2.3.41}$$

Hence, applying Lemma 2.3.5.1 on Q_1, we find a $\tilde{v} \in W_\beta^{1,2}(Q_1;\mathbb{R}^n)$ and a subsequence, which we denote $(v_j)_{j\in\Lambda_1}$ for an infinite set $\Lambda_1 \subset \mathbb{N}$, which satisfies $v_j \rightharpoonup \tilde{v}$ in $W_\beta^{1,2}(Q_1;\mathbb{R}^n)$, $v_j \to \tilde{v}$ in $L_\beta^2(Q_1;\mathbb{R}^n)$ and almost everywhere as $j \to \infty$ with $j \in \Lambda_1$. Notice that $(v_j)_{j\in\Lambda_1}$ converges weakly to v in $W_\beta^{1,2}(Q_1;\mathbb{R}^n)$ because $(v_j)_{j\in\mathbb{N}}$ does and so, by the uniqueness of weak limits, we deduce $\tilde{v} = v$ in Q_1. Hence $v_j \to v$ in $L_\beta^2(Q_1;\mathbb{R}^n)$ and almost everywhere as $j \to \infty$ as well.

Similarly, we observe that the sequence $(v_j)_{j\in\Lambda_1}$ is a bounded sequence in $W_\beta^{1,2}(Q_2;\mathbb{R}^n)$ in view of (2.3.41) and so we apply Lemma 2.3.5.1 to this sequence. We obtain a subsequence $(v_j)_{j\in\Lambda_2}$, where $\Lambda_2 \subset \Lambda_1$ is an infinite set, of $(v_j)_{j\in\Lambda_1}$ which satisfies all the properties of this sequence and, in addition, converges to

v in $L^2_\beta(Q_2;\mathbb{R}^n)$ and almost everywhere in Q_2. Inductively, for every $i \in \mathbb{N}$, we obtain sequences $(v_j)_{j\in\Lambda_i}$ with $\Lambda_{i+1} \subset \Lambda_i$ such that $(v_j)_{j\in\Lambda_i}$ converges to v in $L^2_\beta(Q_i;\mathbb{R}^n)$ and almost everywhere in Q_i.

Now we extract a diagonal-type subsequence from the collection of sequences $\{(v_j)_{j\in\Lambda_i} : i \in \mathbb{N}\}$ with additional bounds on the $L^2_\beta(Q_i;\mathbb{R}^n)$ distance from the terms in the sequence to v for each i. Since $(v_j)_{j\in\Lambda_i}$ converges to v in $L^2_\beta(Q_i;\mathbb{R}^n)$ and $\Lambda_{i+1} \subset \Lambda_i$ for each $i \in \mathbb{N}$, we can choose an increasing sequence of numbers $(k_i)_{i\in\mathbb{N}}$ with $k_i \in \Lambda_i$ such that

$$\int_{Q_i} |v_k - v|^2 \mathrm{d}\mu_\beta < \frac{\int_{\frac{1}{i+1}}^{\frac{1}{i}} x_{m+1}^\beta \mathrm{d}x}{2^i} \leq \frac{1}{i^{2+\beta}2^i} \tag{2.3.42}$$

for $k \geq k_i$. Then the sequence $(v_{k_i})_{i\in\mathbb{N}}$ converges to v almost everywhere in $Q_1^+(0)$ and in $L^2_\beta(Q_k;\mathbb{R}^n)$ for all $k \in \mathbb{N}$ as $i \to \infty$. We claim that $v_{k_i} \to v$ in $L^2_\beta(Q_1^+(0);\mathbb{R}^n)$ as $i \to \infty$.

We write the L^2_β distance from v_{k_i} to v as an integral over two regions in $Q_1^+(0)$, depending on i; fix $i \in \mathbb{N}$ and consider

$$\int_{Q_1^+(0)} |v_{k_i} - v|^2 \mathrm{d}\mu_\beta = \int_{Q'\times\left(0,\frac{1}{i+1}\right)} |v_{k_i} - v|^2 \mathrm{d}\mu_\beta + \int_{Q_i} |v_{k_i} - v|^2 \mathrm{d}\mu_\beta. \tag{2.3.43}$$

By (2.3.42) we have $\int_{Q_i} |v_{k_i} - v| \mathrm{d}\mu_\beta < \frac{1}{i^{2+\beta}2^i}$. We will show a similar bound, in terms of i, for the quantity $\int_{Q'\times\left(0,\frac{1}{i+1}\right)} |v_{k_i} - v| \mathrm{d}\mu_\beta$.

It follows from the discussion in section 2.3.4, applied to the functions $x_{m+1} \mapsto \int_{Q'} |v_{k_i}(x', x_{m+1}) - v(x', x_{m+1})|^2 \mathrm{d}x'$ and combined with Fubini's Theorem, that we may choose $c_i \in \left(\frac{1}{1+i}, \frac{1}{i}\right)$ such that

$$\int_{Q'} |v_{k_i}(x', c_i) - v(x', c_i)|^2 \mathrm{d}x' \leq \frac{1}{\left(\int_{\frac{1}{i+1}}^{\frac{1}{i}} x_{m+1}^\beta \mathrm{d}x_{m+1}\right)} \int_{\frac{1}{i+1}}^{\frac{1}{i}} \int_{Q'} |v_{k_i} - v|^2 \mathrm{d}\mu_\beta. \tag{2.3.44}$$

Now for each $i \in \mathbb{N}$, we calculate

$$\begin{aligned} \int_0^{\frac{1}{i+1}} \int_{Q'} |v_{k_i} - v|^2 \mathrm{d}\mu_\beta \leq{}& 4\int_0^{\frac{1}{i+1}} \int_{Q'} |v_{k_i} - v_{k_i}(x', c_i)|^2 \mathrm{d}\mu_\beta \\ &+ 4\int_0^{\frac{1}{i+1}} \int_{Q'} |v - v(x', c_i)|^2 \mathrm{d}\mu_\beta \\ &+ 4\int_0^{\frac{1}{i+1}} \int_{Q'} |v_{k_i}(x', c_i) - v(x', c_i)|^2 \mathrm{d}\mu_\beta. \end{aligned} \tag{2.3.45}$$

To complete the proof we bound each of the terms on the right hand side of (2.3.45) in terms of i. We proceed with the term

$$\int_0^{\frac{1}{i+1}} \int_{Q'} |v - v(x', c_i)|^2 \mathrm{d}\mu_\beta,$$

noting that the bound for the same integral with v replaced by v_{k_i} is identical. We write this integral in terms of the derivative of v with respect to the $m+1$th variable and apply Hölder's inequality to see that

$$\begin{aligned}
&\int_0^{\frac{1}{i+1}} \int_{Q'} |v(x', x_{m+1}) - v(x', c_i)|^2 \mathrm{d}\mu_\beta \\
&\leq \int_0^{\frac{1}{i+1}} \int_{Q'} \left(\int_{x_{m+1}}^{c_i} \frac{\partial v}{\partial x_{m+1}}(x', s) \mathrm{d}s \right)^2 \mathrm{d}\mu_\beta \\
&= \int_0^{\frac{1}{i+1}} x_{m+1}^\beta \int_{Q'} \left(\int_{x_{m+1}}^{c_i} s^{-\frac{\beta}{2}} s^{\frac{\beta}{2}} \frac{\partial v}{\partial x_{m+1}}(x', s) \mathrm{d}s \right)^2 \mathrm{d}x' \mathrm{d}x_{m+1} \\
&\leq \frac{c_i^{1-\beta}}{1-\beta} \int_0^{\frac{1}{i+1}} x_{m+1}^\beta \int_{Q'} \int_0^{c_i} s^\beta \left| \frac{\partial v}{\partial x_{m+1}} \right|^2 (x', s) \mathrm{d}s \mathrm{d}x' \mathrm{d}x_{m+1} \\
&\leq \frac{c_i^{1-\beta}}{1-\beta^2} c_i^{1+\beta} \int_0^{c_i} \int_{Q'} \left| \frac{\partial v}{\partial x_{m+1}} \right|^2 \mathrm{d}\mu_\beta \\
&\leq \frac{1}{1-\beta^2} \frac{1}{i^2} M^2. \qquad (2.3.46)
\end{aligned}$$

We apply (2.3.44) followed by (2.3.42) to see that

$$\begin{aligned}
\int_0^{\frac{1}{i+1}} \int_{Q'} |v_{k_i}(x', c_i) - v(x', c_i)|^2 \mathrm{d}\mu_\beta &\leq \frac{\int_0^{\frac{1}{i+1}} x_{m+1}^\beta \mathrm{d}x_{m+1}}{\int_{\frac{1}{i+1}}^{\frac{1}{i}} x_{m+1}^\beta \mathrm{d}x} \int_{\frac{1}{i+1}}^{\frac{1}{i}} \int_{Q'} |v_{k_i} - v|^2 \mathrm{d}\mu_\beta \\
&\leq \frac{\int_0^{\frac{1}{i+1}} x_{m+1}^\beta \mathrm{d}x_{m+1}}{2^i} \\
&< \frac{1}{(1+\beta) i^{1+\beta} 2^i}. \qquad (2.3.47)
\end{aligned}$$

Finally we combine (2.3.42), (2.3.43),(2.3.45), (2.3.46) and (2.3.47) to see that

$$\int_{Q_1^+(0)} |v_{k_i} - v|^2 \mathrm{d}\mu_\beta \leq \frac{1}{i^{2+\beta} 2^i} + \frac{4}{(1+\beta) i^{1+\beta} 2^i} + \frac{8}{1-\beta^2} \frac{1}{i^2} M^2 \to 0 \text{ as } i \to \infty$$

which concludes the proof. $\square$

2.4 Properties of Solutions of $\mathbf{div}(x_{m+1}^{\beta}\nabla v) = 0$

The properties of harmonic functions with respect to the Euclidean metric, solutions to Laplace's equation, play an important role in the theory of some geometric semi-linear partial differential equations where the Laplace operator is the highest order term. The variational problem for E^{β} that we will consider gives rise to a highest order term of the form $\operatorname{div}(x_{m+1}^{\beta}\nabla v)$. We therefore expect solutions of (2.0.1), accompanied with Dirichlet or Neumann boundary data, to be useful for our analysis.

At various points in this section, we will want to apply results for single partial differential equations to a system of equations. In the situations we consider, this is permitted because we are considering solutions of a system where the equations for each component are the same. This means that the systems de-couple and we could consider the theory for each of the components of the solution instead. We do not remark on this any further.

Whenever we consider (2.0.1) on a domain Ω whose boundary intersects $\partial\mathbb{R}^{m+1}_{+}$, we will prescribe zero Neumann type data on $\partial\Omega \cap \partial\mathbb{R}^{m+1}_{+}$. As we will see shortly, this permits the even reflection of a solution in $\partial\mathbb{R}^{m+1}_{+}$. Thus we may either consider (2.0.1) with prescribed Neumann-type data or we could consider solutions of

$$\operatorname{div}(|x_{m+1}|^{\beta}\nabla v) = \sum_{i=1}^{m+1} \frac{\partial}{\partial x_i}\left(|x_{m+1}|^{\beta}\frac{\partial v}{\partial x_i}\right) = 0 \tag{2.4.1}$$

in open $\Omega \subset \mathbb{R}^{m+1}$. Generally we will be dealing with weak solutions of (2.4.1): for $\Omega \subset \mathbb{R}^{m+1}$ open, we say that $v \in W^{1,2}_{\beta}(\Omega;\mathbb{R}^n)$ is a weak solution of (2.4.1) if

$$\int_{\Omega} |x_{m+1}|^{\beta} \langle \nabla v, \nabla \phi \rangle \,\mathrm{d}x = \sum_{i=1}^{m+1} \int_{\Omega} |x_{m+1}|^{\beta} \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial \phi}{\partial x_i} \right\rangle \mathrm{d}x = 0 \tag{2.4.2}$$

for every $\phi \in C^{\infty}_{0}(\Omega;\mathbb{R}^n)$. We have expanded the expression for the measure $\mathrm{d}\mu_{\beta}$ into its constituent parts to emphasise the fact that the weight $|x_{m+1}|^{\beta}$ is the coefficient of the partial differential equation. This notation will be common throughout.

Remark 2.4.0.1. The functional $\phi \mapsto \int_{\Omega} |x_{m+1}|^{\beta} \langle \nabla v, \nabla \phi \rangle \,\mathrm{d}x$ is continuous on $C^{\infty}_{0}(\Omega;\mathbb{R}^n)$ which is dense in $W^{1,2}_{\beta,0}(\Omega;\mathbb{R}^n)$ and hence, by approximation,

$$\int_{\Omega} |x_{m+1}|^{\beta} \langle \nabla v, \nabla \phi \rangle \,\mathrm{d}x = 0 \tag{2.4.3}$$

for every $\phi \in W^{1,2}_{\beta,0}(\Omega;\mathbb{R}^n)$ if v is a weak solution of (2.4.1).

If Ω satisfies $\overline{\Omega} \subset \mathbb{R}^{m+1}\backslash(\mathbb{R}^m \times \{0\})$ then (2.4.1) is a uniformly elliptic second order partial differential equation and hence any weak solution is smooth by the regularity theory in Chapter 8 of [21]. Furthermore every weak solution of (2.4.1) is smooth on any open Ω with $\Omega \subset \mathbb{R}^{m+1}_+$ or $\Omega \subset \mathbb{R}^{m+1}_- = \mathbb{R}^m \times (-\infty, 0)$, since such an Ω can be written as a union of bounded domains with closure contained in $\mathbb{R}^{m+1}_+$ or $\mathbb{R}^{m+1}_-$. This reduces regularity questions about solutions of (2.4.1) to domains overlapping $\mathbb{R}^m \times \{0\}$. Next we introduce some function spaces and notation to facilitate further discussion of the solutions to (2.4.1) in such domains.

2.4.1 Spaces of Smooth and Continuous Functions

We have already encountered the space $C^\infty(\Omega;\mathbb{R}^n)$ of smooth, $\mathbb{R}^n$-valued functions on open $\Omega \subset \mathbb{R}^{m+1}$, and the subset $C^\infty_0(\Omega;\mathbb{R}^n)$ of $C^\infty(\Omega;\mathbb{R}^n)$, comprised of smooth functions with compact support in Ω. In order to discuss solutions and their derivatives, we will also require spaces of continuous and differentiable functions.

Let $\Omega \subset \mathbb{R}^{m+1}$ be open, $k \in \mathbb{N}$, $\gamma \in (0,1]$ and $\alpha = (\alpha_1, \ldots, \alpha_{m+1}) \in \mathbb{N}^{m+1} \cup \{0\}$ denote a multi-index. We denote the αth partial derivative, weak or classical, of a map $v : \Omega \to \mathbb{R}^n$, by

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_{m+1}} x_{m+1}}$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_{m+1}$.

The following function spaces are introduced in section 4.1 of [21] and, as stated there, the spaces we can equip with a norm are all Banach spaces. The space of continuous functions $v : \Omega \to \mathbb{R}^n$ is denoted $C(\Omega;\mathbb{R}^n)$. The space of bounded, uniformly continuous functions on $\overline{\Omega}$ is $C(\overline{\Omega};\mathbb{R}^n)$ which, if Ω is bounded, may be given the norm $||v||_{C(\overline{\Omega};\mathbb{R}^n)} = \sup_\Omega |v|$. We write $C^k(\Omega;\mathbb{R}^n)$ for the space of k times continuously differentiable $v : \Omega \to \mathbb{R}^n$. The space of k times differentiable functions whose first k derivatives all have continuous extensions to $\overline{\Omega}$ is denoted $C^k(\overline{\Omega};\mathbb{R}^n)$. If Ω is bounded, this space may be given the norm $||v||_{C^k(\overline{\Omega};\mathbb{R}^n)} = \sum_{j=0}^k \sup_{|\alpha|=j} ||D^\alpha v||_{C(\overline{\Omega};\mathbb{R}^n)}$. Let $C^{0,\gamma}(\Omega;\mathbb{R}^n)$ denote the space of locally Hölder continuous, if $\gamma \in (0,1)$, or locally Lipschitz continuous, if $\gamma = 1$, functions $v : \Omega \to \mathbb{R}^n$. The space of Hölder or Lipschitz continuous functions on Ω is denoted $C^{0,\gamma}(\overline{\Omega};\mathbb{R})$ for $\gamma \in (0,1)$ and $\gamma = 1$ respectively. When Ω is bounded, this space may be given the norm $||v||_{C^{0,\gamma}(\overline{\Omega};\mathbb{R}^n)} = ||v||_{C(\overline{\Omega};\mathbb{R}^n)} + [v]_{C^{0,\gamma}(\overline{\Omega};\mathbb{R}^n)}$, where

$$[v]_{C^{0,\gamma}(\overline{\Omega};\mathbb{R}^n)} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|v(x)-v(y)|}{|x-y|^\gamma} \tag{2.4.4}$$

is a semi-norm on $C^{0,\gamma}(\overline{\Omega};\mathbb{R}^n)$.

2.4.2 Solutions of div$(x_{m+1}^\beta \nabla v) = 0$ with Neumann-type Boundary Data

When we consider a solution v of (2.0.1) on an open $\Omega \subset \mathbb{R}_+^{m+1}$ with $\partial\Omega \cap \partial\mathbb{R}_+^{m+1} \neq \emptyset$, we will usually stipulate that it satisfies a Neumann-type boundary condition on this part of the boundary. This defines a Neumann-type problem where, explicitly, we require

$$\begin{aligned} \operatorname{div}(x_{m+1}^\beta \nabla v) &= 0 \text{ in } \Omega \\ \lim_{x_{m+1}\to 0^+} x_{m+1}^\beta \frac{\partial v}{\partial x_{m+1}} &= 0 \text{ in } \partial^0\Omega \end{aligned} \tag{2.4.5}$$

where $\partial^0\Omega$ is defined as the interior, with respect to $\partial\mathbb{R}_+^{m+1}$, of $\partial\Omega \cap \partial\mathbb{R}_+^{m+1}$. The weak formulation of this problem is as follows; we say that $v \in W_\beta^{1,2}(\Omega;\mathbb{R}^n)$ is a weak solution of the Neumann problem (2.4.5) in Ω if

$$\int_\Omega x_{m+1}^\beta \langle \nabla v, \nabla \phi \rangle \,\mathrm{d}x = \sum_{i=1}^{m+1} \int_\Omega x_{m+1}^\beta \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial \phi}{\partial x_i} \right\rangle \mathrm{d}x = 0 \tag{2.4.6}$$

for every $\phi \in \mathcal{D}_+(\Omega;\mathbb{R}^n) = \{\phi = \psi|_\Omega : \psi \in C_0^\infty(\hat{\Omega};\mathbb{R}^n)\}$ where $\hat{\Omega} = \{x = (x', x_{m+1}) \in \mathbb{R}^{m+1} : (x', |x_{m+1}|) \in \Omega \cup \partial^0\Omega\}$ is the union of Ω, its reflection in $\partial\mathbb{R}_+^{m+1}$ and $\partial^0\Omega$.

Remark 2.4.2.1. The functional $\phi \mapsto \int_\Omega |x_{m+1}|^\beta \langle \nabla v, \nabla \phi \rangle \,\mathrm{d}x$ is continuous on $\mathcal{D}_+(\Omega;\mathbb{R}^n)$ which is dense in its completion, denoted $H_\beta^{1,+}(\Omega;\mathbb{R}^n)$, with respect to the $W_\beta^{1,2}(\Omega;\mathbb{R}^n)$ norm. This completion is a closed subspace of $W_\beta^{1,2}(\Omega;\mathbb{R}^n)$, since the dense subspace in $H_\beta^{1,+}(\Omega;\mathbb{R}^n)$ is contained in the dense subspace in $W_\beta^{1,2}(\Omega;\mathbb{R}^n)$, and thus a reflexive Banach space in its own right. Hence, by approximation,

$$\int_\Omega |x_{m+1}|^\beta \langle \nabla v, \nabla \phi \rangle \,\mathrm{d}x = 0 \tag{2.4.7}$$

for every $\phi \in H_\beta^{1,+}(\Omega;\mathbb{R}^n)$.

We will now show that a weak solution of (2.4.5) in a domain $\Omega \subset \mathbb{R}_+^{m+1}$, where $\partial^0\Omega \neq \emptyset$, can be evenly reflected in $\mathbb{R}^m \times \{0\}$ to give a weak solution of (2.4.1) in $\hat{\Omega}$. The next Lemma shows that the even reflection of a Sobolev function, defined on Ω, in $\mathbb{R}^m \times \{0\}$ gives a Sobolev function in the reflected domain $\hat{\Omega}$ and, furthermore, a symmetric function on this domain restricts to a Sobolev function on Ω.

Lemma 2.4.2.1. *Suppose $\Omega \subset \mathbb{R}^{m+1}_+$ with $\partial^0\Omega \neq \emptyset$. If $v \in W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ or $v \in H^{1,+}_\beta(\Omega;\mathbb{R}^n)$ then the even reflection $\hat{v}(x', x_{m+1}) = v(x', |x_{m+1}|)$, for $(x', x_{m+1}) \in \hat{\Omega}$, is in $W^{1,2}_\beta(\hat{\Omega};\mathbb{R}^n)$ or $W^{1,2}_{\beta,0}(\hat{\Omega};\mathbb{R}^n)$ respectively. Conversely, if $\hat{v} \in W^{1,2}_\beta(\hat{\Omega};\mathbb{R}^n)$ or $\hat{v} \in W^{1,2}_{\beta,0}(\hat{\Omega};\mathbb{R}^n)$ satisfies $\hat{v}(x', x_{m+1}) = \hat{v}(x', -x_{m+1})$ for every $(x', x_{m+1}) \in \hat{\Omega}$, then $v = \hat{v}|_\Omega \in W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ or $v \in H^{1,+}_\beta(\Omega;\mathbb{R}^n)$ respectively. The relationship between the weak derivatives of v and $\hat{v}$ is given by*

$$\frac{\partial \hat{v}}{\partial x_i}(x', x_{m+1}) = \frac{\partial v}{\partial x_i}(x', |x_{m+1}|) \tag{2.4.8}$$

for $i = 1, \ldots, m$ and

$$\frac{\partial \hat{v}}{\partial x_{m+1}}(x', x_{m+1}) = sgn(x_{m+1})\frac{\partial v}{\partial x_{m+1}}(x', |x_{m+1}|). \tag{2.4.9}$$

Proof. We prove the Lemma by constructing appropriate approximating sequences of smooth functions. Our calculations are essentially independent of the Sobolev spaces under consideration so we assume that we are working with elements of $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ and $W^{1,2}_\beta(\hat{\Omega};\mathbb{R}^n)$. The calculations for elements of $H^{1,+}_\beta(\Omega;\mathbb{R}^n)$ and $W^{1,2}_{\beta,0}(\hat{\Omega};\mathbb{R}^n)$ are almost identical.

Suppose that $\hat{v} \in W^{1,2}_\beta(\hat{\Omega};\mathbb{R}^n)$ and let $(\hat{\phi}_k)_{k\in\mathbb{N}}$, with each $\hat{\phi}_k$ contained in the dense subspace of smooth functions in $W^{1,2}_\beta(\hat{\Omega};\mathbb{R}^n)$, be an approximating sequence such that $||\hat{v} - \hat{\phi}_k||_{W^{1,2}_\beta(\hat{\Omega};\mathbb{R}^n)} \to 0$ as $k \to \infty$. Since $||v - \hat{\phi}_k||_{W^{1,2}_\beta(\Omega;\mathbb{R}^n)} \leq ||\hat{v} - \hat{\phi}_k||_{W^{1,2}_\beta(\hat{\Omega};\mathbb{R}^n)}$ we deduce that $(\hat{\phi}_k)|_\Omega \to v$ as $k \to \infty$ in $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ and hence $v \in W^{1,2}_\beta(\Omega;\mathbb{R}^n)$.

Now we assume $v \in W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ and consider the even reflection $\hat{v}(x', x_{m+1}) = v(x', |x_{m+1}|)$ defined for $(x', x_{m+1}) \in \hat{\Omega}$. To construct an approximating sequence of smooth functions for $\hat{v}$, we first consider the reflection of a smooth approximating sequence for v. Choose $(\phi_k)_{k\in\mathbb{N}}$, with ϕ_k contained in the dense subspace of smooth elements in $W^{1,2}_\beta(\Omega;\mathbb{R}^n)$, such that $||v - \phi_k||_{W^{1,2}_\beta(\Omega;\mathbb{R}^n)} \to 0$ as $k \to \infty$. Define $\hat{\phi}_k(x', x_{m+1}) = \phi_k(x', |x_{m+1}|)$ for $(x', x_{m+1}) \in \hat{\Omega}$. We show that $\hat{\phi}_k(x', x_{m+1}) \in W^{1,2}_\beta(\hat{\Omega};\mathbb{R}^n)$.

We calculate

$$\frac{\partial \hat{\phi}_k}{\partial x_i}(x', x_{m+1}) = \frac{\partial \phi_k}{\partial x_i}(x', |x_{m+1}|) \tag{2.4.10}$$

and

$$\frac{\partial \hat{\phi}_k}{\partial x_{m+1}}(x', x_{m+1}) = \mathrm{sgn}(x_{m+1}) \frac{\partial \phi_k}{\partial x_{m+1}}(x', |x_{m+1}|) \tag{2.4.11}$$

for $x_{m+1} \neq 0$ using the chain rule. Next we verify that this relationship defines the weak derivatives of $\hat{\phi}_k$ in $\hat{\Omega}$.

First suppose $i = 1, \ldots, m$, let $\Omega_- = \hat{\Omega} \backslash \Omega$ and let $\psi \in C_0^\infty(\hat{\Omega}; \mathbb{R}^n)$. We have, using the change of variables $x_{m+1} \mapsto -x_{m+1}$,

$$\begin{aligned} \int_{\hat{\Omega}} \frac{\partial \hat{\phi}_k}{\partial x_i} \psi \mathrm{d}x &= \int_\Omega \frac{\partial \phi_k}{\partial x_i} \psi \mathrm{d}x + \int_{\Omega_-} \frac{\partial \hat{\phi}_k}{\partial x_i} \psi \mathrm{d}x \\ &= \int_\Omega \frac{\partial \phi_k}{\partial x_i} \psi \mathrm{d}x + \int_\Omega \frac{\partial \phi_k}{\partial x_i} \psi(x', -x_{m+1}) \mathrm{d}x. \end{aligned} \tag{2.4.12}$$

Since ϕ_k is smooth on Ω and ψ is smooth on $\hat{\Omega}$ and vanishes on a compact subset of $\hat{\Omega}$ we may integrate by parts with respect to x_i. This gives

$$\begin{aligned} \int_\Omega \frac{\partial \phi_k}{\partial x_i} \psi \mathrm{d}x &= -\int_\Omega \phi_k \frac{\partial \psi}{\partial x_i} \mathrm{d}x \\ &= -\int_\Omega \hat{\phi}_k \frac{\partial \psi}{\partial x_i} \mathrm{d}x \end{aligned} \tag{2.4.13}$$

and, combining the integration by parts with the change of variable $x_{m+1} \mapsto -x_{m+1}$ again,

$$\begin{aligned} \int_\Omega \frac{\partial \phi_k}{\partial x_i} \psi(x', -x_{m+1}) \mathrm{d}x &= -\int_\Omega \phi_k \frac{\partial \psi}{\partial x_i}(x', -x_{m+1}) \mathrm{d}x \\ &= -\int_{\Omega_-} \hat{\phi}_k \frac{\partial \psi}{\partial x_i} \mathrm{d}x. \end{aligned} \tag{2.4.14}$$

Combining (2.4.12), (2.4.13) and (2.4.14) shows that the weak partial derivatives $\frac{\partial \hat{\phi}_k}{\partial x_i}$ are defined by (2.4.10). Now suppose $i = m+1$. Observe that $\hat{\phi}_k$ is continuous in $\hat{\Omega}$ and smooth in $\hat{\Omega} \backslash \overline{\partial^0 \Omega}$. We calculate

$$\int_{\hat{\Omega}} \frac{\partial \hat{\phi}_k}{\partial x_{m+1}} \psi \mathrm{d}x = \int_\Omega \frac{\partial \phi_k}{\partial x_{m+1}} \psi \mathrm{d}x - \int_{\Omega_-} \frac{\partial \phi_k}{\partial x_{m+1}}(x', -x_{m+1}) \psi \mathrm{d}x. \tag{2.4.15}$$

An integration by parts yields

$$\int_\Omega \frac{\partial \phi_k}{\partial x_{m+1}} \psi \mathrm{d}x = -\int_{\partial^0 \Omega} \phi_k \psi \mathrm{d}x' - \int_\Omega \phi_k \frac{\partial \psi}{\partial x_{m+1}} \mathrm{d}x. \tag{2.4.16}$$

Moreover, a change of variables, via the map $x_{m+1} \mapsto -x_{m+1}$, and an integration by parts gives

$$
\begin{aligned}
-\int_{\Omega_-} \frac{\partial \phi_k}{\partial x_{m+1}}(x', -x_{m+1})\psi \mathrm{d}x &= -\int_{\Omega} \frac{\partial \phi_k}{\partial x_{m+1}} \psi(x', -x_{m+1}) \mathrm{d}x \\
&= \int_{\partial^0\Omega} \phi_k \psi \mathrm{d}x' - \int_{\Omega} \phi_k \frac{\partial \psi}{\partial x_{m+1}}(x', -x_{m+1}) \mathrm{d}x \\
&= \int_{\partial^0\Omega} \phi_k \psi \mathrm{d}x' - \int_{\Omega_-} \hat{\phi}_k \frac{\partial \psi}{\partial x_{m+1}} \mathrm{d}x. \qquad (2.4.17)
\end{aligned}
$$

Combining (2.4.15), (2.4.16) and (2.4.17) we see that the weak derivative $\frac{\partial \hat{\phi}_k}{\partial x_{m+1}}$ is defined by (2.4.11). Furthermore, we have

$$
\int_{\hat{\Omega}} |x_{m+1}|^\beta |\nabla \hat{\phi}_k|^2 \mathrm{d}x = 2\int_{\Omega} x_{m+1}^\beta |\nabla \phi_k|^2 \mathrm{d}x < \infty \qquad (2.4.18)
$$

and

$$
\int_{\hat{\Omega}} |x_{m+1}|^\beta |\hat{\phi}_k|^2 \mathrm{d}x = 2\int_{\Omega} x_{m+1}^\beta |\phi_k|^2 \mathrm{d}x < \infty \qquad (2.4.19)
$$

and so $\hat{\phi}_k \in W_\beta^{1,2}(\hat{\Omega}; \mathbb{R}^n)$ for every k.

Now we construct an approximating sequence of smooth functions in $W_\beta^{1,2}(\hat{\Omega}; \mathbb{R}^n)$ for $\hat{v}$. We deduce, using the inequalities (2.4.18) and (2.4.19), that

$$
||\hat{v} - \hat{\phi}_k||_{W_\beta^{1,2}(\hat{\Omega};\mathbb{R}^n)} = 2^{\frac{1}{2}} ||v - \phi_k||_{W_\beta^{1,2}(\Omega;\mathbb{R}^n)} \to 0 \text{ as } k \to \infty.
$$

Furthermore, since each $\hat{\phi}_k \in W_\beta^{1,2}(\hat{\Omega}; \mathbb{R}^n)$, each $\hat{\phi}_k$ has an approximating sequence of smooth functions and we may choose an increasing sequence of numbers $(j_k)_{k\in\mathbb{N}}$ and smooth functions ϕ_{j_k} in the dense set in $W_\beta^{1,2}(\hat{\Omega}; \mathbb{R}^n)$ such that

$$
||\hat{\phi}_k - \phi_{j_k}||_{W_\beta^{1,2}(\hat{\Omega};\mathbb{R}^n)} \leq \frac{1}{k}.
$$

It follows that

$$
||\hat{v} - \phi_{j_k}||_{W_\beta^{1,2}(\hat{\Omega};\mathbb{R}^n)} \leq ||\hat{v} - \hat{\phi}_k||_{W_\beta^{1,2}(\Omega;\mathbb{R}^n)} + ||\phi_{j_k} - \hat{\phi}_k||_{W_\beta^{1,2}(\Omega;\mathbb{R}^n)} \to 0 \text{ as } k \to \infty
$$

and hence $(\phi_{j_k})_{k\in\mathbb{N}}$ is the required approximating sequence for $\hat{v}$.

The equalities (2.4.8) and (2.4.9) hold since they hold for $\hat{\phi}_k$ and $\hat{\phi}_k \to \hat{v}$ in $W_\beta^{1,2}(\hat{\Omega}; \mathbb{R}^n)$ and so, extracting subsequences if necessary, we conclude that $\hat{\phi}_k \to \hat{v}$ and $\frac{\partial \hat{\phi}_k}{\partial x_i} \to \frac{\partial \hat{v}}{\partial x_i}$, for $i = 1, \ldots, m+1$, almost everywhere. □

Lemma 2.4.2.2. *Let* $\Omega \subset \mathbb{R}_+^{m+1}$ *with* $\partial\Omega \cap \partial\mathbb{R}_+^{m+1} \neq \emptyset$ *and let* $\hat{\Omega} = \{x = (x', x_{m+1}) \in \mathbb{R}^{m+1} : (x', |x_{m+1}|) \in \Omega \cup \partial^0\Omega\}$. *The even reflection, with respect to*

$\mathbb{R}^m \times \{0\}$, *of a weak solution of* (2.4.5) *in* $\Omega \subset \mathbb{R}^{m+1}_+$ *is a weak solution of* (2.4.1) *in* $\hat{\Omega}$. *Conversely, a weak solution of* (2.4.1)*, which is symmetric with respect to* $\mathbb{R}^m \times \{0\}$, *in* $\hat{\Omega}$ *is a weak solution of* (2.4.5) *in* Ω.

Proof. Suppose that two measurable functions $v : \Omega \to \mathbb{R}^n$ and $\hat{v} : \hat{\Omega} \to \mathbb{R}^n$ are related via

$$\hat{v}(x', x_{m+1}) = \begin{cases} v(x', x_{m+1}) & (x', x_{m+1}) \in \Omega \\ v(x', -x_{m+1}) & (x', x_{m+1}) \in \Omega_- = \hat{\Omega} \backslash \Omega \end{cases}$$

It follows from Lemma 2.4.2.1 that $v \in W^{1,2}_\beta(\Omega; \mathbb{R}^n)$ if, and only if, $\hat{v} \in W^{1,2}_\beta(\hat{\Omega}; \mathbb{R}^n)$.

Henceforth, we assume v and $\hat{v}$ are related as above. We have

$$\begin{aligned} \int_{\hat{\Omega}} |x_{m+1}|^\beta \langle \nabla \hat{v}, \nabla \phi \rangle \, \mathrm{d}x = & \int_\Omega x^\beta_{m+1} \langle \nabla v, \nabla \phi \rangle \, \mathrm{d}x \\ & + \int_{\Omega_-} (-x_{m+1})^\beta \langle \nabla \hat{v}, \nabla \phi \rangle \, \mathrm{d}x. \end{aligned} \tag{2.4.20}$$

Furthermore, we write the integral over Ω_- as

$$\begin{aligned} \int_{\Omega_-} (-x_{m+1})^\beta \langle \nabla \hat{v}, \nabla \phi \rangle \, \mathrm{d}x = & \int_{\Omega_-} (-x_{m+1})^\beta \langle \nabla' \hat{v}, \nabla' \phi \rangle \, \mathrm{d}x \\ & + \int_{\Omega_-} (-x_{m+1})^\beta \left\langle \frac{\partial \hat{v}}{\partial x_{m+1}}, \frac{\partial \phi}{\partial x_{m+1}} \right\rangle \mathrm{d}x \end{aligned} \tag{2.4.21}$$

where ∇' is the gradient with respect to $x_1, \ldots, x_m$ and, for any functions f, h such that $\nabla' f, \nabla' h$ exist, $\langle \nabla' f, \nabla' h \rangle = \sum_{i=1}^m \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial h}{\partial x_i} \right\rangle$. Combining the expressions for the weak derivatives for v and $\hat{v}$ given by Lemma 2.4.2.1 with the change of variables $x_{m+1} \mapsto -x_{m+1}$ we calculate

$$\int_{\Omega_-} (-x_{m+1})^\beta \langle \nabla' \hat{v}, \nabla' \phi \rangle \, \mathrm{d}x = \int_\Omega x^\beta_{m+1} \langle \nabla' v, \nabla' \phi(x', -x_{m+1}) \rangle \, \mathrm{d}x \tag{2.4.22}$$

and

$$\begin{aligned} & \int_{\Omega_-} (-x_{m+1})^\beta \left\langle \frac{\partial \hat{v}}{\partial x_{m+1}}, \frac{\partial \phi}{\partial x_{m+1}} \right\rangle \mathrm{d}x \\ & = - \int_{\Omega_-} (-x_{m+1})^\beta \left\langle \frac{\partial v}{\partial x_{m+1}}(x', -x_{m+1}), \frac{\partial \phi}{\partial x_{m+1}} \right\rangle \mathrm{d}x \\ & = - \int_\Omega x^\beta_{m+1} \left\langle \frac{\partial v}{\partial x_{m+1}}, \frac{\partial \phi}{\partial x_{m+1}}(x', -x_{m+1}) \right\rangle \mathrm{d}x. \end{aligned} \tag{2.4.23}$$

Together, (2.4.20), (2.4.21), (2.4.22) and (2.4.23) give

$$\begin{aligned}\int_{\hat{\Omega}} |x_{m+1}|^\beta \langle \nabla \hat{v}, \nabla \phi \rangle \,\mathrm{d}x = &\int_\Omega x_{m+1}^\beta \langle \nabla v, \nabla \phi \rangle \,\mathrm{d}x \\ &+ \int_\Omega x_{m+1}^\beta \langle \nabla' v, \nabla' \phi(x', -x_{m+1}) \rangle \,\mathrm{d}x \\ &- \int_\Omega x_{m+1}^\beta \left\langle \frac{\partial v}{\partial x_{m+1}}, \frac{\partial \phi}{\partial x_{m+1}}(x', -x_{m+1}) \right\rangle \mathrm{d}x. \quad (2.4.24)\end{aligned}$$

Now we show that if $\hat{v}$ is a weak solution of (2.4.1), that is $\operatorname{div}(|x_{m+1}|^\beta \nabla \hat{v}) = 0$ in $\hat{\Omega}$ in the weak sense, then v is a weak solution of (2.4.5), or in other words $\operatorname{div}(x_{m+1}^\beta \nabla v) = 0$ in Ω and $x_{m+1}^\beta \frac{\partial v}{\partial x_{m+1}} = 0$ in $\partial^0 \Omega$ in the weak sense, and vice versa.

Suppose v is a weak solution of (2.4.5). Then v satisfies

$$\int_\Omega x_{m+1}^\beta \langle \nabla v, \nabla \phi \rangle \,\mathrm{d}x = 0$$

for every $\phi \in \mathcal{D}_+(\Omega; \mathbb{R}^n)$. We want to show that $\hat{v}$ satisfies (2.4.2) for every $\phi \in C_0^\infty(\hat{\Omega}; \mathbb{R}^n)$. Choose such a ϕ and define $\psi(x', x_{m+1}) = \phi(x', -x_{m+1})$. It follows that $\phi|_\Omega$ and $\psi|_\Omega$ are in $\mathcal{D}_+(\Omega; \mathbb{R}^n)$. Furthermore, their derivatives are related by

$$\frac{\partial \phi}{\partial x_i}(x', x_{m+1}) = \frac{\partial \psi}{\partial x_i}(x', -x_{m+1}) \tag{2.4.25}$$

for $i = 1, \ldots, m$ and

$$\frac{\partial \phi}{\partial x_{m+1}}(x', x_{m+1}) = -\frac{\partial \psi}{\partial x_{m+1}}(x', -x_{m+1}). \tag{2.4.26}$$

Hence, using (2.4.24), (2.4.25) and (2.4.26), we have

$$\begin{aligned}\int_{\hat{\Omega}} |x_{m+1}|^\beta \langle \nabla \hat{v}, \nabla \phi \rangle \,\mathrm{d}x = &\int_\Omega x_{m+1}^\beta \langle \nabla v, \nabla \phi \rangle \,\mathrm{d}x \\ &+ \int_\Omega x_{m+1}^\beta \langle \nabla' v, \nabla' \phi(x', -x_{m+1}) \rangle \,\mathrm{d}x \\ &- \int_\Omega x_{m+1}^\beta \left\langle \frac{\partial v}{\partial x_{m+1}}, \frac{\partial \phi}{\partial x_{m+1}}(x', -x_{m+1}) \right\rangle \mathrm{d}x \\ = &\int_\Omega x_{m+1}^\beta \langle \nabla v, \nabla \phi \rangle \,\mathrm{d}x \\ &+ \int_\Omega x_{m+1}^\beta \langle \nabla v, \nabla \psi \rangle \,\mathrm{d}x \\ = &\ 0 \end{aligned} \tag{2.4.27}$$

since v is a weak solution of (2.4.5) and $\phi|_\Omega$ and $\psi|_\Omega$ are in $\mathcal{D}_+(\Omega;\mathbb{R}^n)$. This shows that $\hat{v}$ is a weak solution of (2.4.1) since $\phi \in C_0^\infty(\hat{\Omega};\mathbb{R}^n)$ was arbitrary.

Conversely, suppose that $\hat{v}$ is a weak solution of (2.4.1). Then for every $\phi \in C_0^\infty(\hat{\Omega};\mathbb{R}^n)$ we have

$$\int_{\hat{\Omega}} |x_{m+1}|^\beta \langle \nabla \hat{v}, \nabla \phi \rangle \, \mathrm{d}x = 0.$$

By approximation, as in remark 2.4.0.1, this holds for every $\phi \in W^{1,2}_{\beta,0}(\hat{\Omega};\mathbb{R}^n)$. Suppose that $\phi \in \mathcal{D}_+(\Omega;\mathbb{R}^n)$. It follows from Lemma 2.4.2.1 that the even reflection $\hat{\phi}$ of ϕ in $\mathbb{R}^m \times \{0\}$ is in $W^{1,2}_{\beta,0}(\hat{\Omega};\mathbb{R}^n)$. Moreover, using the relationship between the derivatives of ϕ and $\hat{\phi}$ given by (2.4.8) and (2.4.9) from Lemma 2.4.2.1, combined with (2.4.24), we see that

$$\begin{aligned} 0 &= \int_{\hat{\Omega}} x_{m+1}^\beta \left\langle \nabla \hat{v}, \nabla \hat{\phi} \right\rangle \mathrm{d}x \\ &= \int_{\Omega} x_{m+1}^\beta \langle \nabla v, \nabla \phi \rangle \, \mathrm{d}x \\ &\quad + \int_{\Omega} x_{m+1}^\beta \left\langle \nabla' v, \nabla' \hat{\phi}(x', -x_{m+1}) \right\rangle \mathrm{d}x \\ &\quad - \int_{\Omega} x_{m+1}^\beta \left\langle \frac{\partial v}{\partial x_{m+1}}, \frac{\partial \hat{\phi}}{\partial x_{m+1}}(x', -x_{m+1}) \right\rangle \mathrm{d}x \\ &= 2 \int_{\Omega} x_{m+1}^\beta \langle \nabla v, \nabla \phi \rangle \, \mathrm{d}x. \end{aligned} \tag{2.4.28}$$

Since this holds for every $\phi \in \mathcal{D}_+(\Omega;\mathbb{R}^n)$ we conclude that v is a weak solution of (2.4.5). This concludes the proof. □

2.4.3 Continuity Properties of Solutions of $\mathbf{div}(|x_{m+1}|^\beta \nabla v) = 0$ on Sets Overlapping $\partial\mathbb{R}^{m+1}_+$

On any $\Omega \subset \mathbb{R}^{m+1}$ with $\Omega \cap \partial\mathbb{R}^{m+1}_+ \neq \emptyset$, as mentioned previously, it follows from the regularity theory for second order linear elliptic equations in Chapter 8 of [21] that a solution of (2.4.1) is smooth in $\Omega \backslash \partial\mathbb{R}^{m+1}_+$. When the ellipticity degenerates, arguments from the theory of degenerate elliptic equations must instead be used to conclude continuity and higher regularity of solutions.

In order to conclude the Hölder continuity of solutions to the variational problems considered later we will need to use a consequence of the continuity of solutions to (2.4.1); a result of Fabes et al [19] shows that a weak solution of (2.4.1) in an $\Omega \subset \mathbb{R}^{m+1}$ with $\partial^0\Omega \neq \emptyset$ is locally Hölder continuous in Ω. We state their result in the context considered here and do not quote the whole result, merely the part that we need.

Lemma 2.4.3.1 ([19] Part of Theorem 2.3.12)**.** *Suppose v is a weak solution of $div(|x_{m+1}|^{\beta}\nabla v) = 0$ in $\Omega \subset \mathbb{R}^{m+1}$. Then v is locally Hölder continuous in Ω.*

Corollary 2.4.3.1. *Let v be a weak solution of $div(|x_{m+1}|^{\beta}\nabla v) = 0$ in $B_R(x_0) \subset \mathbb{R}^{m+1}$. There exists a positive constant C and a $\gamma = \gamma(m,\beta) \in (0,1)$ such that*

$$|v(x) - v(y)| \leq C|x-y|^{\gamma}$$

for $x, y \in B_{\frac{R}{2}}(x_0)$.

Since the equation $\mathrm{div}(|x_{m+1}|^{\beta}\nabla v) = 0$ is linear we deduce the following reverse Poincaré inequality, analogously to the case of linear uniformly elliptic equations.

Lemma 2.4.3.2. *Let v be a weak solution of (2.4.1) in a ball $B_R(x_0) \subset \mathbb{R}^{m+1}$ and let $r \leq R$. Then*

$$\int_{B_{\frac{r}{2}}(x_0)} |\nabla v|^2 \mathrm{d}\mu_{\beta} \leq \frac{C}{r^2}\int_{B_r(x_0)} |v-\lambda|^2 \mathrm{d}\mu_{\beta} \tag{2.4.29}$$

for any $\lambda \in \mathbb{R}^n$ and a positive constant C.

Proof. Let $\eta \in C_0^{\infty}(B_r(x_0);\mathbb{R}^n)$ be a cutoff function with $\eta \equiv 1$ in $B_{\frac{r}{2}}(x_0)$ and $0 \leq \eta \leq 1$. Furthermore, suppose that $|\nabla\eta| \leq \frac{C}{r}$ for a fixed positive $C > 2$. Although we are discussing PDEs here, since the weight $|x_{m+1}|^{\beta}$ does not play a significant role we use the notation $\mathrm{d}\mu_{\beta}$ in place of $|x_{m+1}|^{\beta}\mathrm{d}x$. We want to test (2.4.2) against $\phi = \eta^2(v-\lambda)$. This is an admissible test function by remark 2.4.0.1 since η is smooth with compact support in $B_r(x_0)$ and $v-\lambda \in W_{\beta}^{1,2}(\Omega;\mathbb{R}^n)$. Hence, we find

$$\int_{B_r(x_0)} \left\langle \nabla v, \nabla(\eta^2(v-\lambda)) \right\rangle \mathrm{d}\mu_{\beta} = 0.$$

We expand this expression to see that

$$\int_{B_r(x_0)} \eta^2|\nabla v|^2 \mathrm{d}\mu_{\beta} = -2\int_{B_r(x_0)} \eta \left\langle \nabla v, \nabla\eta \cdot (v-\lambda)\right\rangle \mathrm{d}\mu_{\beta}. \tag{2.4.30}$$

Using Young's inequality, $ab \leq \delta\frac{a^2}{2} + \frac{b^2}{2\delta}$ for $a, b \geq 0$ and $\delta > 0$, we see that

$$2\int_{B_r(x_0)} \eta \left\langle \nabla v, \nabla\eta \cdot (v-\lambda)\right\rangle \mathrm{d}\mu_\beta \leq \delta \int_{B_r(x_0)} \eta^2 |\nabla v|^2 \mathrm{d}\mu_\beta + \frac{C}{\delta}\int_{B_r(x_0)} |\nabla\eta|^2 |v-\lambda|^2 \mathrm{d}\mu_\beta. \qquad (2.4.31)$$

Choosing $\delta = \frac{1}{2}$ and combining (2.4.30) and (2.4.31) we see that

$$\int_{B_r(x_0)} \eta^2 |\nabla v|^2 \mathrm{d}\mu_\beta \leq C\int_{B_r(x_0)} |\nabla\eta|^2 |v-\lambda|^2 \mathrm{d}\mu_\beta$$

and since $|\nabla\eta| \leq \frac{C}{r}$ this yields (2.4.29) as required. □

Combining Corollary 2.4.3.1 and Lemma 2.4.3.2 we deduce the following result which we will require in our regularity theory later on.

Corollary 2.4.3.2. *Let v be a weak solution of (2.4.5) in a half-ball $B_R^+(x_0)$ with $x_0 \in \mathbb{R}^m \times \{0\}$. There exists a $\gamma = \gamma(m,\beta) \in (0,1)$ and a positive constant C such that*

$$\left(\frac{r}{2}\right)^{1-m-\beta} \int_{B^+_{\frac{r}{2}}(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \leq Cr^{2\gamma} \qquad (2.4.32)$$

for every $r \leq \frac{R}{2}$.

Proof. The even reflection $\hat{v}$ of v in $\partial\mathbb{R}^{m+1}_+$ is a weak solution of $\mathrm{div}(|x_{m+1}|^\beta \nabla\hat{v}) = 0$ in $B_R(x_0)$ by Lemma 2.4.2.2. Therefore we may apply Lemma 2.4.3.2 to $\hat{v}$. We do so, choosing $\lambda = \hat{v}(x_0)$ in the lemma, to see that

$$\int_{B_{\frac{r}{2}}(x_0)} |x_{m+1}|^\beta |\nabla\hat{v}|^2 \mathrm{d}x \leq \frac{C}{r^2}\int_{B_r(x_0)} |x_{m+1}|^\beta |\hat{v} - \hat{v}(x_0)|^2 \mathrm{d}x \qquad (2.4.33)$$

for every $r \leq \frac{R}{2}$. Moreover, corollary 2.4.3.1 gives a $\gamma \in (0,1)$ such that

$$|\hat{v} - \hat{v}(x_0)| \leq Cr^\gamma \qquad (2.4.34)$$

for a constant C and, since $2\int_{B^+_{\frac{r}{2}}(x_0)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x = \int_{B_{\frac{r}{2}}(x_0)} |x_{m+1}|^\beta |\nabla\hat{v}|^2 \mathrm{d}x$, we deduce the claim of the corollary by combining (2.4.33) and (2.4.34). □

Chapter 3

Hölder Continuity of Energy Minimisers

First we set the scene. Let $\mathbb{R}^{m+1}_+ = \mathbb{R}^m \times (0, \infty)$ and consider the Riemannian metric g on $\mathbb{R}^{m+1}_+$ defined in Euclidean coordinates by

$$g(x)(\cdot, \cdot) = x^\alpha_{m+1} \sum_{i,j=1}^{m+1} \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j, \tag{3.0.1}$$

where $\alpha \in \mathbb{R}$ is fixed and $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The matrix representing g has elements $g_{ij}(x) = x^\alpha_{m+1}\delta_{ij}$ and the elements of the inverse are given by

$$g^{ij}(x) = x^{-\alpha}_{m+1}\delta_{ij}.$$

We also calculate

$$\det(g(x)) = x^{(m+1)\alpha}_{m+1}.$$

If we try to extend g by continuity to take values on $\mathbb{R}^m \times \{0\}$ then the metric would become degenerate or singular depending on the sign of α. If $\alpha > 0$ then the extension would be identically 0 along $\mathbb{R}^m \times \{0\}$ and if $\alpha < 0$ then the extension would be infinite.

We define the Dirichlet energy, or simply energy, of a map $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ by

$$\begin{aligned} E^\beta(v) =& \frac{1}{2} \int_{\mathbb{R}^{m+1}_+} |\nabla v|^2_g \sqrt{\det(g)} \mathrm{d}x \\ =& \frac{1}{2} \int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \end{aligned} \tag{3.0.2}$$

where $\beta = \alpha(\frac{m+1}{2} - 1) = \frac{\alpha(m-1)}{2}$ and

$$|\nabla v|_g^2 = \sum_{i,j=1}^{m+1} g^{ij} \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle = x_{m+1}^{-\alpha} \sum_{i=1}^{m+1} \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle.$$

Recall the assumptions on m, β as discussed in Remark 2.2.1.1. We will study maps with image in a smooth, compact Riemannian manifold N. The Nash embedding theorem [36] guarantees that we may isometrically embed N in $\mathbb{R}^n$ for some $n \in \mathbb{N}$ and we assume this is the case henceforth. For technical reasons we also assume, translating N if necessary, that $0 \in N$.

Define

$$\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N) = \{v \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; \mathbb{R}^n) : v(x) \in N \text{ for almost every } x \in \mathbb{R}^{m+1}_+\}. \tag{3.0.3}$$

This space of functions is non-empty since $0 \in N$. The regularity properties of a particular class of critical points $v \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N)$ of E^{β} with respect to smooth variations, both of the dependent variable $v \in N$ and of the independent variable $x \in \mathbb{R}^{m+1}_+$, are our primary concern. Before discussing our main questions regarding these critical points, we will first derive two systems of partial differential equations which they necessarily solve.

3.1 Critical Points of E^{β}

We formulate all the results in this section in anticipation of a connection to a family of variational problems, corresponding to those we will consider for the E^{β}, for the boundary values of functions in $\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N)$ on open $\mathcal{O} \subset \partial\mathbb{R}^{m+1}_+$. The classes of admissible variations of E^{β} will be chosen accordingly. Henceforth $\mathcal{O}$ denotes an open subset of $\partial\mathbb{R}^{m+1}_+$.

3.1.1 The Euler-Lagrange Equations

Here we consider critical points $v \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; N)$ of E^{β} with respect to smooth variations of the dependent variable which leave the boundary values of v unchanged outside $\mathcal{O}$. To construct such a variation we will compactly perturb the map v by adding a small multiple of a smooth function $\psi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ with $\psi(\cdot, 0) \in C_0^{\infty}(\mathcal{O}; \mathbb{R}^n)$ to v. In general, for any $s > 0$ the sum $v + s\psi$ will not have values in N. In order to define a variation which does have values in N we make use of the nearest point projection onto N, defined as follows. Since N is compact, theorem 1 in section 2.12.3 of [46] gives a tubular neighbourhood of N, which has the form $U_\delta(N) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, N) < \delta\}$ for a $\delta = \delta(N) > 0$, and a smooth map $\pi_N : U_\delta(N) \to N$ such that $|\pi_N(y) - y| = \mathrm{dist}(y, N)$ for every $y \in U_\delta(N)$. For sufficiently small s we define a variation of v by

$$v_s = \pi_N(v + s\psi) \in N.$$

We say v is a critical point of E^β with respect to smooth variations of the dependent variable relative to $\mathcal{O}$ if

$$\left.\frac{d}{ds}\right|_{s=0} E^\beta(v_s) = 0$$

for every variation of the form v_s. In order to calculate $\left.\frac{d}{ds}\right|_{s=0} E^\beta(v_s)$ explicitly we follow [33] section 3.2. Observe that

$$|\nabla v_s|^2 = \sum_{i=1}^{m+1} \left\langle D\pi_N(v + s\psi)\left(\frac{\partial v}{\partial x_i} + s\frac{\partial \psi}{\partial x_i}\right), D\pi_N(v + s\psi)\left(\frac{\partial v}{\partial x_i} + s\frac{\partial \psi}{\partial x_i}\right)\right\rangle \tag{3.1.1}$$

where $D\pi_N$ is the derivative of the nearest point projection. Differentiating both sides of (3.1.1) with respect to s yields

$$\begin{aligned}
&\frac{d}{ds}\left(x_{m+1}^\beta |\nabla v_s|^2\right) \\
=&2x_{m+1}^\beta \sum_{i=1}^{m+1} \left\langle D\pi_N(v + s\psi)\left(\frac{\partial v}{\partial x_i} + s\frac{\partial \psi}{\partial x_i}\right), \mathrm{Hess}\pi_N(v + s\psi)\left(\frac{\partial v}{\partial x_i} + s\frac{\partial \psi}{\partial x_i}, \psi\right)\right\rangle \\
&+ \ 2x_{m+1}^\beta \sum_{i=1}^{m+1} \left\langle D\pi_N(v + s\psi)\left(\frac{\partial v}{\partial x_i} + s\frac{\partial \psi}{\partial x_i}\right), D\pi_N(v + s\psi)\left(\frac{\partial \psi}{\partial x_i}\right)\right\rangle.
\end{aligned} \tag{3.1.2}$$

Lemma 3.1 of [33] shows that $D\pi_N(y)$ is the orthogonal projection onto T_yN and hence, using the fact that $\frac{\partial v}{\partial x_i}$ is tangential to N, we evaluate (3.1.2) at $s = 0$ to see that

$$\left.\frac{d}{ds}\right|_{s=0}\left(x_{m+1}^{\beta}|\nabla v_s|^2\right) = 2x_{m+1}^{\beta}\sum_{i=1}^{m+1}\left\langle D\pi_N(v)\left(\frac{\partial v}{\partial x_i}\right), \mathrm{Hess}\pi_N(v)\left(\frac{\partial v}{\partial x_i},\psi\right)\right\rangle$$
$$+ 2x_{m+1}^{\beta}\sum_{i=1}^{m+1}\left\langle D\pi_N(v)\left(\frac{\partial v}{\partial x_i}\right), D\pi_N(v)\left(\frac{\partial \psi}{\partial x_i}\right)\right\rangle$$
$$= 2x_{m+1}^{\beta}\sum_{i=1}^{m+1}\left\langle \frac{\partial v}{\partial x_i}, \mathrm{Hess}\pi_N(v)\left(\frac{\partial v}{\partial x_i},\psi\right) + \frac{\partial \psi}{\partial x_i}\right\rangle. \quad (3.1.3)$$

We split the vector ψ into its normal part $\psi^{\perp}$ and tangential part $\psi^{\top}$ with respect to T_vN, so that $\psi = \psi^{\perp} + \psi^{\top}$. It follows from lemma 3.2 in [33] that

$$\sum_{i=1}^{m+1}\left\langle \frac{\partial v}{\partial x_i}, \mathrm{Hess}\pi_N(v)\left(\frac{\partial v}{\partial x_i},\psi\right)\right\rangle = \sum_{i=1}^{m+1}\left\langle \frac{\partial v}{\partial x_i}, \mathrm{Hess}\pi_N(v)\left(\frac{\partial v}{\partial x_i},\psi^{\top}+\psi^{\perp}\right)\right\rangle$$
$$= -\sum_{i=1}^{m+1}\left\langle \psi^{\perp}, A(v)\left(\frac{\partial v}{\partial x_i},\frac{\partial v}{\partial x_i}\right)\right\rangle$$

where A, a section of $T^*N\otimes T^*N\otimes(TN)^{\perp}$, is the second fundamental form of N. Since $A(v(x))$ is a normal vector for almost every x we find

$$\sum_{i=1}^{m+1}\left\langle \frac{\partial v}{\partial x_i}, \mathrm{Hess}\pi_N(v)\left(\frac{\partial v}{\partial x_i},\psi\right)\right\rangle = -\sum_{i=1}^{m+1}\left\langle \psi, A(v)\left(\frac{\partial v}{\partial x_i},\frac{\partial v}{\partial x_i}\right)\right\rangle. \quad (3.1.4)$$

We combine (3.1.3) and (3.1.4) to see that

$$\left.\frac{d}{ds}\right|_{s=0} E^{\beta}(v_s) = \sum_{i=1}^{m+1}\int_{\mathbb{R}_+^{m+1}} x_{m+1}^{\beta}\left(\left\langle \frac{\partial v}{\partial x_i},\frac{\partial \psi}{\partial x_i}\right\rangle - \left\langle \psi, A(v)\left(\frac{\partial v}{\partial x_i},\frac{\partial v}{\partial x_i}\right)\right\rangle\right)\mathrm{d}x. \quad (3.1.5)$$

If $v\in \dot{W}_{\beta}^{1,2}(\mathbb{R}_+^{m+1};N)$ is a critical point of E^{β} with respect to smooth variations of the dependent variable relative to $\mathcal{O}$ then, in view of (3.1.5), v satisfies

$$\sum_{i=1}^{m+1}\int_{\mathbb{R}_+^{m+1}} x_{m+1}^{\beta}\left(\left\langle \frac{\partial v}{\partial x_i},\frac{\partial \psi}{\partial x_i}\right\rangle - \left\langle \psi, A(v)\left(\frac{\partial v}{\partial x_i},\frac{\partial v}{\partial x_i}\right)\right\rangle\right)\mathrm{d}x = 0 \quad (3.1.6)$$

for every $\psi\in\mathcal{D}_+(\mathbb{R}_+^{m+1};\mathbb{R}^n)$ with $\psi(\cdot,0)\in C_0^{\infty}(\mathcal{O};\mathbb{R}^n)$. We abbreviate (3.1.6) to

$$\int_{\mathbb{R}_+^{m+1}} x_{m+1}^{\beta}\left(\langle\nabla v,\nabla\psi\rangle - \langle\psi, A(v)(\nabla v,\nabla v)\rangle\right)\mathrm{d}x = 0 \quad (3.1.7)$$

where

$$\langle \nabla v, \nabla \psi \rangle = \sum_{i=1}^{m+1} \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial \psi}{\partial x_i} \right\rangle \quad \text{and} \quad A(v)\left(\nabla v, \nabla v\right) = \sum_{i=1}^{m+1} A(v) \left(\frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_i} \right).$$

It is important to note that the Euler Lagrange equations (3.1.6) contain Neumann type boundary condition for v on $\mathcal{O}$. We interpret this as follows: assuming v is as smooth as necessary, we integrate by parts in (3.1.6) to obtain

$$\begin{aligned} &\sum_{i=1}^{m+1} \int_{\mathbb{R}^{m+1}_+} \left\langle \frac{\partial}{\partial x_i} \left(x_{m+1}^{\beta} \frac{\partial v}{\partial x_i} \right) + x_{m+1}^{\beta} A(v) \left(\frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_i} \right), \psi \right\rangle \mathrm{d}x \\ &= - \int_{\mathcal{O}} \left\langle \left(x_{m+1}^{\beta} \frac{\partial v}{\partial x_{m+1}} \right) \Big|_{x_{m+1}=0}, \psi(x', 0) \right\rangle \mathrm{d}x' \end{aligned} \tag{3.1.8}$$

which holds for every $\psi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ with $\psi(\cdot, 0) \in C_0^\infty(\mathcal{O}; \mathbb{R}^n)$. If $\psi \in C_0^\infty(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ we deduce

$$\sum_{i=1}^{m+1} \left(\frac{\partial}{\partial x_i} \left(x_{m+1}^{\beta} \frac{\partial v}{\partial x_i} \right) + x_{m+1}^{\beta} A(v) \left(\frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_i} \right) \right) = 0 \quad \text{in } \mathbb{R}^{m+1}_+. \tag{3.1.9}$$

We abbreviate (3.1.9) to

$$\operatorname{div}(x_{m+1}^{\beta} \nabla v) + x_{m+1}^{\beta} A(v)\left(\nabla v, \nabla v\right) = 0 \quad \text{in } \mathbb{R}^{m+1}_+. \tag{3.1.10}$$

Furthermore, if v is sufficiently smooth in $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$, the Neumann type boundary condition in (3.1.8) becomes

$$\lim_{x_{m+1} \to 0^+} x_{m+1}^{\beta} \frac{\partial v}{\partial x_{m+1}} = 0 \text{ in } \mathcal{O}. \tag{3.1.11}$$

A map satisfying (3.1.9) is said to be *harmonic*. Furthermore, a harmonic map satisfying (3.1.11) is said to be *harmonic with respect to the Neumann type boundary condition* (3.1.11). A map satisfying (3.1.6) for every $\psi \in C_0^\infty(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ is *weakly harmonic* and a map satisfying (3.1.6) for every $\psi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ with $\psi(\cdot, 0) \in C_0^\infty(\mathcal{O}; \mathbb{R}^n)$ is *weakly harmonic with respect to the Neumann type boundary condition* (3.1.11).

3.1.2 The Stationary Equations

A weakly harmonic map may be discontinuous in general. In [38] Rivière establishes the existence of everywhere discontinuous weakly harmonic maps from the unit ball $B^n \subset \mathbb{R}^n$ to the unit sphere $\mathbb{S}^p \subset \mathbb{R}^{p+1}$ for $n \geq 3$ and $p \geq 2$. However, there are various regularity theories, which we will discuss shortly, for weakly harmonic maps satisfying additional conditions. The least stringent of these is that a weakly harmonic map must also be a critical point of the energy with respect to smooth variations of the independent variable $x \in \mathbb{R}^{m+1}_+$.

Weakly harmonic maps with respect to the Neumann-type boundary condition (3.1.11) are weakly harmonic by definition and so may also be discontinuous in $\mathbb{R}^{m+1}_+$. Such maps may also have discontinuities on $\mathcal{O}$, at least if $\beta = 0$ [23], as discussed subsequently in Section 3.2. Thus we stipulate that such a map must also be a critical point of E^β with respect to smooth variations of the independent variable which leave its boundary values unchanged outside $\mathcal{O}$. Such critical points are solutions of another system of PDEs in addition to (3.1.10) and we derive these presently.

Recall that a $\phi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^{m+1})$ is the restriction to $\mathbb{R}^{m+1}_+$ of a smooth, compactly supported function ψ defined on $\mathbb{R}^{m+1}$. Without relabelling, we also write ϕ to mean $\psi|_{\overline{\mathbb{R}^{m+1}_+}}$.

Define

$$\Phi_s(x) = x + s\phi(x)$$

for $x \in \overline{\mathbb{R}^{m+1}_+}$, where $\phi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^{m+1})$ is such that

$$\phi(\cdot, 0) \in C_0^\infty(\mathcal{O}; \partial\mathbb{R}^{m+1}_+)$$

and $|s|$ is small enough to make Φ_s into a diffeomorphism of $\overline{\mathbb{R}^{m+1}_+}$ with $\Phi_s(\mathcal{O}) \subset \mathcal{O}$. Then, for $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ we define a variation of v given by $v_s = v \circ \Phi_s$ which is also a map in $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$.

A $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ is said to be a critical point of E^β with respect to smooth variations of the independent variable relative to $\mathcal{O}$ if $\frac{d}{ds}\big|_{s=0} E^\beta(v_s) = 0$ for any v_s defined as above. We proceed to calculate $\frac{d}{ds}\big|_{s=0} E^\beta(v_s)$. Observe that

$$\begin{aligned} E^\beta(v_s) = {} & \frac{1}{2}\int_{\mathbb{R}^{m+1}_+} x_{m+1}^\beta \sum_{i=1}^{m+1} \left\langle \frac{\partial v}{\partial x_i}(\Phi_s), \frac{\partial v}{\partial x_i}(\Phi_s) \right\rangle \mathrm{d}x \\ & + s \int_{\mathbb{R}^{m+1}_+} x_{m+1}^\beta \sum_{i=1}^{m+1} \left\langle \frac{\partial v}{\partial x_i}(\Phi_s), \sum_{k=1}^{m+1} \frac{\partial v}{\partial x_k}(\Phi_s) \frac{\partial \phi_k}{\partial x_i} \right\rangle \mathrm{d}x \\ & + \frac{1}{2}s^2 \int_{\mathbb{R}^{m+1}_+} x_{m+1}^\beta \sum_{i=1}^{m+1} \left\langle \sum_{k=1}^{m+1} \frac{\partial v}{\partial x_k}(\Phi_s) \frac{\partial \phi_k}{\partial x_i}, \sum_{j=1}^{m+1} \frac{\partial v}{\partial x_j}(\Phi_s) \frac{\partial \phi_j}{\partial x_i} \right\rangle \mathrm{d}x. \end{aligned} \tag{3.1.12}$$

To further our calculation we exploit the fact that Φ_s is a diffeomorphism, with inverse Φ_s^{-1}, and transform

$$\int_{\mathbb{R}_+^{m+1}} x_{m+1}^{\beta} \sum_{i=1}^{m+1} \left\langle \frac{\partial v(\Phi_s)}{\partial x_i}, \frac{\partial v(\Phi_s)}{\partial x_i} \right\rangle \mathrm{d}x$$

in (3.1.12). We see that

$$\begin{aligned}
&\int_{\mathbb{R}_+^{m+1}} x_{m+1}^{\beta} \sum_{i=1}^{m+1} \left\langle \frac{\partial v(\Phi_s)}{\partial x_i}, \frac{\partial v(\Phi_s)}{\partial x_i} \right\rangle \mathrm{d}x \\
=&\int_{\mathbb{R}_+^{m+1}} (\Phi_s^{-1})_{m+1}^{\beta} \sum_{i=1}^{m+1} \left\langle \frac{\partial v(x)}{\partial x_i}, \frac{\partial v(x)}{\partial x_i} \right\rangle |\det D\Phi_s^{-1}| \mathrm{d}x \\
=&\int_{\mathbb{R}_+^{m+1}} (\Phi_s^{-1})_{m+1}^{\beta} \sum_{i=1}^{m+1} \left\langle \frac{\partial v(x)}{\partial x_i}, \frac{\partial v(x)}{\partial x_i} \right\rangle (1 - s\mathrm{div}\phi(x) + O(s^2)) \mathrm{d}x \qquad (3.1.13)
\end{aligned}$$

where D means derivative, we have used the series expansion for $\det D\Phi_s^{-1}$ and the absolute value sign has been omitted since $\det D\Phi_s^{-1}$ is posivite for sufficiently small s. We will need to calculate $\frac{\partial}{\partial s}\big|_{s=0} (\Phi_s^{-1})$ and briefly digress to do this now. The notation $\Phi_s(x) = \Phi(s,x)$ makes the following calculation more transparent. By definition $\Phi(s, \Phi^{-1}(s,x)) = x$ for all $x \in \mathbb{R}_+^{m+1}$ and for all s sufficiently small. Furthermore, $\Phi_0 = \Phi_0^{-1} = Id_{\mathbb{R}_+^{m+1}}$, the identity map on $\mathbb{R}_+^{m+1}$. Let D_x represent the derivative with respect to x. Then $D_x\Phi(s,x) = D_x(x+s\phi) = Id_{R_+^{m+1}} + sD_x\phi$. Thus we calculate

$$\begin{aligned}
\frac{\partial}{\partial s}\bigg|_{s=0} \Phi(s, \Phi^{-1}(s,x)) =& \frac{\partial \Phi}{\partial s}(0, \Phi^{-1}(0,x)) + D_x\Phi|_{(0,\Phi^{-1}(0,x))} \left(\frac{\partial \Phi^{-1}}{\partial s}(0,x) \right) \\
=& \frac{\partial \Phi}{\partial s}(0,x) + \frac{\partial \Phi^{-1}}{\partial s}(0,x) \\
=&0.
\end{aligned}$$

As a result we have

$$\frac{\partial \Phi^{-1}}{\partial s}(0,x) = -\frac{\partial \Phi}{\partial s}(0,x) = -\phi(x). \qquad (3.1.14)$$

Substituting (3.1.13) into (3.1.12), differentiating with respect to s and using

(3.1.14) we see that,

$$
\begin{aligned}
2\left.\frac{d}{ds}\right|_{s=0} E^{\beta}(v_s) = &- \int_{\mathbb{R}^{m+1}_+} \beta x_{m+1}^{\beta-1}\phi_{m+1} \sum_{i=1}^{m+1} \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle \mathrm{d}x \\
&- \int_{\mathbb{R}^{m+1}_+} x_{m+1}^{\beta} \sum_{i=1}^{m+1} \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle (\mathrm{div}\phi)\mathrm{d}x \\
&+ 2\int_{\mathbb{R}^{m+1}_+} x_{m+1}^{\beta} \sum_{i=1}^{m+1} \left\langle \frac{\partial v}{\partial x_i}, \sum_{k=1}^{m+1} \frac{\partial v}{\partial x_k}\frac{\partial \phi_k}{\partial x_i} \right\rangle \mathrm{d}x.
\end{aligned}
\tag{3.1.15}
$$

Thus a critical point of the Dirichlet energy corresponding to variations of the independent variable relative to $\mathcal{O}$ satisfies

$$
\begin{aligned}
&\int_{\mathbb{R}^{m+1}_+} \sum_{i=1}^{m+1}\sum_{k=1}^{m+1} x_{m+1}^{\beta} \left(2\left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_k} \right\rangle - \delta_{ik}|\nabla v|^2 \right) \frac{\partial \phi_k}{\partial x_i}\mathrm{d}x \\
&\qquad = \int_{\mathbb{R}^{m+1}_+} \beta x_{m+1}^{\beta-1}\phi_{m+1}|\nabla v|^2\mathrm{d}x.
\end{aligned}
\tag{3.1.16}
$$

A weakly harmonic map which satisfies (3.1.16) for every $\phi \in C_0^\infty(\mathbb{R}^{m+1}_+;\mathbb{R}^{m+1})$ is said to be *weakly stationary harmonic* or stationary harmonic. A weakly harmonic map with respect to the Neumann type boundary condition (3.1.11) which satisfies (3.1.16) for every $\phi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^{m+1})$ with $\phi(\cdot,0) \in C_0^\infty(\mathcal{O};\partial\mathbb{R}^{m+1}_+)$ is called *weakly stationary harmonic, or stationary harmonic, with respect to the Neumann type boundary condition* (3.1.11).

If we assume v is smooth we may integrate by parts in (3.1.16) to obtain a divergence form equation. We consider (3.1.16) for $\phi \in C_0^\infty(\mathbb{R}^{m+1}_+;\mathbb{R}^{m+1})$ and analyse the term involving $|\nabla v|^2$ on the left hand side. We calculate

$$
\begin{aligned}
\int_{\mathbb{R}^{m+1}_+} \sum_{i=1}^{m+1}\sum_{k=1}^{m+1} x_{m+1}^{\beta}\delta_{ik}|\nabla v|^2 \frac{\partial \phi_k}{\partial x_i}\mathrm{d}x &= \int_{\mathbb{R}^{m+1}_+} \sum_{k=1}^{m+1} \frac{\partial}{\partial x_k}\left(x_{m+1}^{\beta}|\nabla v|^2\phi_k \right)\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^{m+1}_+} \sum_{k=1}^{m+1} \frac{\partial}{\partial x_k}\left(x_{m+1}^{\beta}|\nabla v|^2 \right)\phi_k\mathrm{d}x \\
&= - \int_{\mathbb{R}^{m+1}_+} \sum_{k=1}^{m+1} \frac{\partial}{\partial x_k}\left(x_{m+1}^{\beta}|\nabla v|^2 \right)\phi_k\mathrm{d}x
\end{aligned}
\tag{3.1.17}
$$

since ϕ has compact support in $\mathbb{R}^{m+1}_+$ and thus it follows from an application of the divergence theorem that $\int_{\mathbb{R}^{m+1}_+} \sum_{k=1}^{m+1} \frac{\partial}{\partial x_k}\left(x_{m+1}^{\beta}|\nabla v|^2\phi_k \right) dx = 0$. An integration by parts in the first term on the left hand side of (3.1.16) combined with (3.1.17) gives

$$\begin{aligned}
&\int_{\mathbb{R}^{m+1}_+} \sum_{k=1}^{m+1} \frac{\partial}{\partial x_k} \left(x^\beta_{m+1} |\nabla v|^2 \right) \phi_k \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^{m+1}_+} \sum_{i=1}^{m+1} \sum_{k=1}^{m+1} \frac{\partial}{\partial x_i} \left(x^\beta_{m+1} 2 \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_k} \right\rangle \right) \phi_k \mathrm{d}x \\
&= \int_{\mathbb{R}^{m+1}_+} \beta x^{\beta-1}_{m+1} \phi_{m+1} |\nabla v|^2 \mathrm{d}x.
\end{aligned}$$

Thus, for $i = 1, \ldots, m$ we have

$$\operatorname{div} \left(x^\beta_{m+1} |\nabla v|^2 e_i - 2 x^\beta_{m+1} \left\langle \frac{\partial v}{\partial x_i}, \nabla v \right\rangle \right) = 0 \text{ in } \mathbb{R}^{m+1}_+ \tag{3.1.18}$$

where we have used the notation

$$\operatorname{div} \left(x^\beta_{m+1} \left\langle \frac{\partial v}{\partial x_i}, \nabla v \right\rangle \right) = \sum_{k=1}^{m+1} \frac{\partial}{\partial x_k} \left(x^\beta_{m+1} \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_k} \right\rangle \right).$$

Similarly we find

$$\operatorname{div} \left(x^\beta_{m+1} |\nabla v|^2 e_{m+1} - 2 x^\beta_{m+1} \left\langle \frac{\partial v}{\partial x_{m+1}}, \nabla v \right\rangle \right) = \beta x^{\beta-1}_{m+1} |\nabla v|^2 \text{ in } \mathbb{R}^{m+1}_+. \tag{3.1.19}$$

3.2 Background Theory and Discussion of the Problem

We are interested in the regularity properties of maps $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ which minimise E^β in the following sense:

Definition 3.2.0.1. Let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$. We say that v is E^β minimising, or energy minimising, in $\mathbb{R}^{m+1}_+$ relative to $\mathcal{O} \subset \partial \mathbb{R}^{m+1}_+$, if for every compact $K \subset \mathbb{R}^{m+1}$ with $K \cap \partial \mathbb{R}^{m+1}_+ \subset \mathcal{O}$ and for every $w \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ with $v|_{\mathbb{R}^{m+1}_+ \setminus K} = w|_{\mathbb{R}^{m+1}_+ \setminus K}$ we have $E^\beta(v) \leq E^\beta(w)$.

The main question we intend to answer is whether, away from a small set of points, minimisers of E^β relative to $\mathcal{O}$ have regular extensions to $\mathcal{O}$.

An energy minimiser relative to $\mathcal{O}$ is weakly stationary harmonic with respect to the Neumann type boundary condition (3.1.11) and therefore weakly stationary harmonic in $\mathbb{R}^{m+1}_+$. Such a map is also energy minimising in $\mathbb{R}^{m+1}_+$ among maps with the same boundary values and hence stationary harmonic on this domain. There is a rich regularity theory for maps $v : M \to N$, with M, N Riemannian manifolds of some order of differentiability, which are minimisers of the Dirichlet energy (1.0.3) or stationary harmonic maps. The regularity of minimisers was first addressed by Morrey, see [31] and references therein. He used them in his solution of the Plateau Problem for manifold valued maps; given a region bounded by a finite number of closed Jordan curves, find a surface of least area whose boundary comprises the curves. He showed that such maps are as smooth as the regularity M and N permit. Later, Schoen and Uhlenbeck considered the regularity of minimisers $v : M \to N$, with M, N compact Riemannian manifolds [44]. They showed minimisers are regular away from a set of Hausdorff dimension at most $\dim M - 3$. Their approach relied heavily on the use of comparison functions, maps which agree with a given minimiser on the boundary of a domain of interest, and allowed them to directly take advantage of the minimising property. To obtain regularity, they used comparison functions which are smooth mollifications of a given minimiser with energy sufficiently small so that the mollification may be projected (using π_N) onto N and modified to agree with minimisers on the boundary of a small ball. Further constructions of comparison maps allowed them to study the singular set of minimisers. Their methods were extended and simplified by Simon [46] in view of a result of Luckhaus [29] which provided a different method of constructing comparison functions. We will also construct comparison maps in this section, basing our approach on that of Simon. The advantage of this approach is that, when $\beta = 0$, it provides a more general compactness result for sequences of minimising harmonic maps, than those obtained by Schoen and Uhlenbeck in [44]. We expect that a similar compactness result holds for minimisers of E^β relative to $\mathcal{O}$, but do not prove the result in this thesis as we are primarily interested in discerning the regularity of such maps.

If one is considering the regularity properties of stationary or even weakly harmonic maps, it is necessary to use a different approach from that for minimisers as it is no longer fruitful to compare the energies of maps in the same way. Hélein proved full regularity for weakly harmonic maps on a 2 dimensional domain [26]. This corresponds to $m = 1$ in the situation considered here and in this case $\beta = \alpha \frac{m-1}{2} \equiv 0$ regardless of the choice of $\alpha \in \mathbb{R}$ in (3.0.1). His method was to construct an optimal tangent frame on N; the purpose of this frame is to allow one to re-write the equation in a form which shows that the Laplacian of a minimiser is more regular than the a-priori L^1 information obtained from the Euler-Lagrange equation. If the target manifold N has enough symmetry, it is possible to write the Euler Lagrange equation in a form sufficient for such gains in regularity without an optimal frame. Evans proved partial regularity of stationary harmonic maps into spheres using this observation to prove the decay of the energy on concentric balls is sufficient to allow one to conclude continuity, and hence higher regularity [17]. Bethuel, see [3], extended these results to general smooth target manifolds using an optimal tangent frame based on the construction of Hélein [26]. More recently Rivière observed that the Euler-Lagrange equation for harmonic maps with two dimensional domain (and more generally any critical point of any elliptic conformally invariant functional) may be re-written so that the maps Laplacian is equal to the product of its gradient with an anti-symmetric factor [39]. This factor may depend on the map and its gradient. Rivière deduced such maps satisfy a conservation law; the divergence of a first order quantity along solutions vanishes. In dimensions two he showed this yields continuity. This method was then extended to maps whose domain is of any dimension by Rivière and Struwe who noted that the harmonic map equation may still be written as a product in the aforementioned form [40]. An application of a well chosen gauge transformation to the gradient then makes the anti-symmetric factor divergence free and one can again deduce decay estimates for the energy sufficient to conclude continuity.

The result of Hélein, see [26], allows us to conclude that, for maps whose domain is 2 dimensional, minimisers of $E^\beta = E^0$ are smooth since such a map satisfies (3.1.6). In fact, as shown in the proof of Theorem 1.2 in [34], we can reflect a minimiser, or stationary critical point, relative to $\mathcal{O}$ of E^0 evenly in $\mathbb{R} \times \{0\}$ to get a harmonic map defined on $\mathbb{R}^2 \backslash ((\mathbb{R} \times \{0\}) \backslash \mathcal{O})$ which is smooth. The restriction of such a map is then smooth on $\mathcal{O}$.

If $\alpha = 0$ in the definition of our metric then $\beta \equiv 0$, regardless of m. In this case we are considering E^0 for maps defined on $\mathbb{R}^{m+1}_+$. Again, we may reflect a minimiser or stationary harmonic map evenly in $\mathbb{R}^m \times \{0\}$, as in the proof of Theorem 1.3 in [34], to get a harmonic map on $\mathbb{R}^{m+1} \backslash ((\mathbb{R}^m \times \{0\}) \backslash \mathcal{O})$. The aforementioned regularity theory then applies in this case. Otherwise, the singularity or degeneracy of the metrics we consider along the boundary hyperplane $\mathbb{R}^m \times \{0\}$ prevents us from using the regularity theory directly to show that a critical point can be extended in the desired way to the boundary hyperplane. Henceforth we will assume that $m \geq 2$ and $\alpha \neq 0$.

So far we have discussed interior regularity theory for harmonic maps. As stated previously, we are specifically interested in the regularity of minimisers of v in $\mathcal{O}$. On this set, minimisers of E^β weakly satisfy the Neumann type boundary condition (3.1.11). This condition also arises in the free boundary problem for harmonic maps; if v is a weakly harmonic map in $\mathbb{R}^{m+1}_+$ and we require $v(\mathcal{O}) \subset N$ then, at least for $\beta = 0$ (see [23] for example), v weakly satisfies the Neumann type boundary condition (3.1.11). We therefore discuss some results from the theory of free boundary problems as well.

A number of regularity results for various classes of harmonic maps $v : M \to N$ satisfying a free boundary condition, with M, N Riemannian manifolds satisfying conditions described below, have been obtained in the literature as part of an investigation into constrained boundary conditions: $v(\mathcal{O}) \subset \Gamma$ for a relatively open submanifold $\mathcal{O} \subset \partial M$ and a submanifold $\Gamma \subset N$. Such a condition implies $\frac{\partial v}{\partial \nu}$, with ν the unit normal to ∂M, is orthogonal to Γ on $\mathcal{O}$ [23]. Neumann boundary conditions, analogous to (3.1.11) with $\beta = 0$, arise when $\Gamma = N$; in this case $\frac{\partial v}{\partial \nu}$ must be both orthogonal and tangential to N hence equal to zero.

Baldes studied the regularity of weakly harmonic maps which satisfy the constraint on $\mathcal{O} \subsetneq \partial M$ with Dirichlet data on the remainder of the boundary [2]. Both M and N are assumed compact with and without boundary respectively. He required that $v(\mathcal{O}) \subset \Gamma$ for a totally geodesic $\Gamma \subset N$ and showed that if the sectional curvature of N is bounded above and the image of S under the Dirichlet data is contained in a sufficiently small (depending on the curvature bound) geodesic ball in N with centre in Γ, then this mixed boundary problem has a solution which is as smooth as the regularity of M and N permits. In particular, the singular set of such maps is empty. Gulliver and Jost also considered weakly harmonic maps v with constrained boundary conditions $v(\partial M) \subset \Gamma$ [23]. Their hypotheses are in terms of a family of convex functions parametrised by points in Γ; in particular they yield continuity, and higher regularity at free boundary points. They also construct an example of a weakly harmonic map with a discontinuity at the free boundary and illustrate that the second fundamental form, in N, of Γ plays a role in the regularity of v at free boundary points.

There are also works which address the regularity of harmonic maps at a free boundary with no geometric assumptions on M, N or Γ; only compactness and some degree of differentiablity of these manifolds are required. However, to compensate for the lack of geometric hypotheses one must stipulate that the maps in question are energy minimising or stationary harmonic and respect the free boundary condition since, as mentioned previously, there are examples of weakly harmonic maps with discontinuities at the free boundary. Such examples imply that weakly harmonic maps satisfying the free boundary condition may have points of discontinuity at the boundary in general and therefore one can, at most, expect partial regularity on the boundary. Since we assume N is compact and smooth, with no other geometric conditions, we also only expect a partial regularity result for minimisers of E^β relative to $\mathcal{O}$.

Two simultaneous works addressed the regularity of minimising harmonic maps satisfying the free boundary condition $v(\mathcal{O}) \subset \Gamma$ which use different methods to obtain partial regularity results. Hardt and Lin considered the free boundary problem for energy minimisers v with $v(\partial M) \subset N$ as well as the constrained problem $v(\partial M) \subset \Gamma$ [24]. Their methods are based on the construction of comparison maps analogous to those of Schoen and Uhlenbeck [44]; Hardt and Lin use these maps, together with a blow up procedure in the domain, to show that near free boundary points the energy decays sufficiently to conclude continuity. They prove that minimisers are regular on $\mathcal{O}$ away from a set of Hausdorff dimension at most $\dim M - 4$ for the free boundary case and at most $\dim M - 3$ for the constrained problem. Moreover, they give examples of minimisers with singular sets in $\mathcal{O}$ which have precisely the stated Hausdorff dimensions, thus showing that the dimension bounds for the singular set are optimal. Duzaar and Steffen also obtained partial regularity for the constrained problem for minimisers with $v(\mathcal{O}) \subset \Gamma$ [16]. They combined the methods of Schoen and Uhlenbeck with a partial reflection of the minimiser across Γ in N to obtain appropriate comparison maps. Using these maps, they showed that minimisers of the constrained problem are as smooth as allowed by M, N away from a subset of $\mathcal{O}$ with vanishing $\dim M - 2$ Hausdorff measure. They subsequently improved their results, see [15], to achieve the same bound as Hardt and Lin, basing their method on the dimension reducing arguments used by Schoen and Uhlenbeck [44].

Scheven considered the regularity of stationary harmonic maps v with respect to the constraint $v(\mathcal{O}) \subset \Gamma$ [41]. His methods are based on a reflection of v across Γ in N combined with a number of arguments used by Bethuel [3]. He showed that stationary harmonic maps satisfying the free boundary condition are smooth away from a singular set of vanishing $\dim M - 2$ dimensional Hausdorff measure.

A common feature in the aforementioned literature on boundary problems is an extension of the maps in question by reflection across Γ in N and/or across $\mathcal{O}$ in M. We will not need a reflection in N since we only consider maps with the free boundary condition $v(\mathcal{O}) \subset N$. In this case we have the Neumann condition (3.1.11) and similarly to [34] we may, if necessary, just consider the even reflection in $\partial\mathbb{R}^{m+1}_+$ of minimisers of E^β relative to $\mathcal{O}$. We also note that in the literature regarding boundary problems with no geometric assumptions on N, the partial regularity results obtained all give the optimal bound on the Hausdorff dimension of the singular set on the free boundary. In this thesis, we are only interested in obtaining regularity away from a singular set and do not consider reducing the maximum dimension of the set we obtain as it turns out to be small enough for our purposes. However, as mentioned previously, we expect our method of constructing comparison maps to yield a compactness result for sequences of minimisers which, in turn, would then permit the analysis of the singular set of minimisers of E^β relative to $\mathcal{O}$.

The main theorem of this section is the following.

Theorem 3.2.0.1. *Suppose $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ is a minimising harmonic map relative to $\mathcal{O} \subset \partial\mathbb{R}^{m+1}_+$. Then there is a relatively closed set $\Sigma \subset \mathbb{R}^{m+1}_+ \cup \mathcal{O}$ of vanishing $(m+\beta-1)$-dimensional Hausdorff measure, with respect to the Euclidean metric, such that $v \in C^{0,\gamma}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma; N)$ for some $\gamma \in (0,1)$. Furthermore, Σ decomposes into $\Sigma_{int} = \Sigma \cap \mathbb{R}^{m+1}_+$ which is relatively closed in $\mathbb{R}^{m+1}_+$ and has Hausdorff dimension $m-2$, and $\Sigma_{bdry} = \Sigma \cap \mathcal{O}$ which is relatively closed in $\mathcal{O}$ and satisfies $\mathcal{H}^{m+\beta-1}(\Sigma_{bdry}) = 0$.*

The part of the theorem corresponding to regularity in $\mathbb{R}^{m+1}_+$ is a direct consequence of the regularity theory of Schoen and Uhlenbeck. To elucidate; a minimising harmonic map $v : \mathbb{R}^{m+1}_+ \to N$ restricts to a minimising harmonic map on any $(m+1)$-dimensional compact subset K of the Riemannian manifold $(\mathbb{R}^{m+1}_+, x^\alpha_{m+1}\delta_{ij})$ with $\alpha \in \mathbb{R}$. Since K is compact this means that the restriction of the metric is smooth and bounded. Let K° denote the interior of K. Theorem II in [44] guarantees that v is smooth in K° with the possible exception of a set which is closed in K° and has $(m-2)$-dimensional Hausdorff measure. We can express $\mathbb{R}^{m+1}_+$ as a countable union $\cup_{i=1}^\infty K_i$ where each K_i is an $(m+1)$-dimensional compact subset of $\mathbb{R}^{m+1}_+$ and $K_i \subset K_{i+1}$ for every $i \in \mathbb{N}$. Then v is smooth in K_i° with the possible exception of a closed subset Σ_i of K_i°, where Σ_i has vanishing $m-2+\delta$-dimensional Hausdorff measure for every $\delta > 0$. We note that $K_i^\circ\backslash\Sigma_i$ is open in K_i° and hence open in $\mathbb{R}^{m+1}_+$. Thus $\cup_i K_i^\circ\backslash\Sigma_i$ is open in $\mathbb{R}^{m+1}_+$ and its complement is therefore closed in $\mathbb{R}^{m+1}_+$. We conclude that $\mathbb{R}^{m+1}_+\backslash \cup_i (K_i^\circ\backslash\Sigma_i) = \cup_i\Sigma_i$ is closed in $\mathbb{R}^{m+1}_+$ and v is smooth in $\mathbb{R}^{m+1}_+\backslash\Sigma_{\text{int}}$ where $\Sigma_{\text{int}} := \cup_{i=1}^\infty\Sigma_i$. Since each Σ_i has vanishing $m-2+\delta$-dimensional Hausdorff measure (for every $\delta > 0$) we conclude that Σ_{int} has the same property. Hence the Hausdorff dimension of Σ_{int} is at most $m-2$. Furthermore, Σ_{int} comprises all points $x \in \mathbb{R}^{m+1}_+$ such that v is not continuous in any neighbourhood of x since, by Theorem 8.5.1 in [27], continuous weakly harmonic maps are smooth.

To show that minimisers of E^β have the desired extension to $\mathcal{O}$ we develop a regularity theory, analogous to that for harmonic maps, at this boundary. Our proof of Hölder continuity contains a number of results which have counterparts in the theory of harmonic maps.

3.3 Energy Monotonicity

Stationary harmonic maps satisfy a monotonicity formula for an appropriately scaled version of the energy over balls with closure in $\mathbb{R}^{m+1}_+$. This property was proved by Schoen and Uhlenbeck for energy minimisers, see [44] Proposition 2.4, and Price, see the remark after Theorem 1 in [37], for stationary harmonic maps. This monotonicity formula is a key ingredient in the proof of Hölder continuity of both stationary and minimising harmonic maps; it contributes to the proof of energy decay estimates sufficient to conclude the desired continuity.

We show that stationary harmonic maps with respect to the Neumann-type boundary condition (3.1.11) satisfy a similar monotonicity formula on half-balls

$$B^+_\rho(y) = \{x \in \mathbb{R}^{m+1}_+ : |x - y| < \rho\}$$

in $\mathbb{R}^{m+1}_+$ with centre y in $\mathcal{O}$ and which satisfy $\overline{\partial^0 B^+_\rho(y)} = \overline{B^m(y)} \subset \mathcal{O}$, where $\partial^0 B^+_\rho(y)$ is the interior, with respect to $\mathbb{R}^m \times \{0\}$, of $\partial B^+_\rho(y) \cap \partial\mathbb{R}^{m+1}_+$. Moreover, we will derive a version of the formula for balls with closure contained in $\mathbb{R}^{m+1}_+$, with a view to determining what factors the constants involved depend upon.

Many of the estimates we consider involve integrals over $\partial B^+_\rho(y) \cap \mathbb{R}^{m+1}_+$ and so we use the following notation. Let $\Omega \subset \mathbb{R}^{m+1}$ and define

$$\partial^+\Omega = \partial\Omega \cap \mathbb{R}^{m+1}_+.$$

We will also use a smooth cutoff function

$$\chi(s) = \begin{cases} 0 & s \in (-\infty, \frac{1}{2}] \\ 0 \le \chi(s) \le 1 & s \in (\frac{1}{2}, 1) \\ 1 & s \in [1, \infty) \end{cases} \tag{3.3.1}$$

in the following proofs.

3.3.1 Boundary Energy Monotonicity

Lemma 3.3.1.1. *Suppose $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ is a weakly stationary harmonic map with respect to the Neumann-type boundary condition* (3.1.11)*. Suppose y*

in $\mathcal{O}$ and consider $B_R^+(y)$ with $\overline{\partial^0 B_R^+(y)} \subset \mathcal{O}$. Then

$$\begin{aligned} & r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^{\beta} |\nabla v|^2 \, \mathrm{d}x - s^{1-m-\beta} \int_{B_s^+(y)} x_{m+1}^{\beta} |\nabla v|^2 \, \mathrm{d}x \\ &= \; 2 \int_{B_r^+(y) \setminus B_s^+(y)} x_{m+1}^{\beta} \frac{|(x-y) \cdot \nabla v|^2}{|x-y|^{m+1+\beta}} \mathrm{d}x \end{aligned} \tag{3.3.2}$$

whenever $0 \leq s \leq r \leq R$ and therefore

$$\rho^{1-m-\beta} \int_{B_\rho^+(y)} x_{m+1}^{\beta} |\nabla v|^2 \, \mathrm{d}x$$

is a non-decreasing function of ρ for $0 < \rho \leq R$.

Proof. We follow [46] Section 2.4 and [33] Lemma 3.3. Our strategy is to test the weak form of the stationary equations (3.1.16) for a map v, which is a stationary harmonic map with respect to the Neumann type boundary condition (3.1.11), against a sequence of functions which are the product of the vector $x - y$ and a smooth, radially symmetric, sequence of real valued functions which converge pointwise to the indicator function of $B_\rho^+(y)$. We take the limit of the resulting integrals to yield the statement of the lemma in terms of the derivatives, with respect to the radius, of the quantities involved and then integrate to give the conclusion.

We test (3.1.16) with $\phi(x) = (x - y)\eta(x)$ where $\eta \in C_0^\infty(B_\rho(y))$, since in this case $\phi(x', 0) = (x' - y', 0)\eta(x) \in \partial\mathbb{R}_+^{m+1}$ for every $x = (x', 0) \in \mathcal{O}$ and so $\phi(\cdot, 0) \in C_0^\infty(\mathcal{O}; \partial\mathbb{R}_+^{m+1})$. This gives

$$\begin{aligned} & \int_{\mathbb{R}_+^{m+1}} \sum_{i=1}^{m+1} \sum_{k=1}^{m+1} x_{m+1}^{\beta} \left(2 \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_k} \right\rangle - \delta_{ik} |\nabla v|^2 \right) \left(\delta_{ik} \eta + (x_k - y_k) \frac{\partial \eta}{\partial x_i} \right) \mathrm{d}x \\ & \qquad = \int_{\mathbb{R}_+^{m+1}} \beta x_{m+1}^{\beta} \eta |\nabla v|^2 \mathrm{d}x \end{aligned} \tag{3.3.3}$$

since $y_{m+1} = 0$. In more concise notation, after rearranging, (3.3.3) becomes

$$\begin{aligned} & (m - 1 + \beta) \int_{\mathbb{R}_+^{m+1}} x_{m+1}^{\beta} |\nabla v|^2 \eta \mathrm{d}x + \int_{\mathbb{R}_+^{m+1}} x_{m+1}^{\beta} (x - y) \cdot \nabla \eta \, |\nabla v|^2 \, \mathrm{d}x \\ & = 2 \int_{\mathbb{R}_+^{m+1}} x_{m+1}^{\beta} \langle (x - y) \cdot \nabla v, \nabla \eta \cdot \nabla v \rangle \, \mathrm{d}x. \end{aligned} \tag{3.3.4}$$

Let χ be the cutoff function defined in (3.3.1). The smooth functions defined by $\eta_j(x) = \chi(j(\rho - |x - y|))$ are admissible choices for η and the sequence η_j

converges pointwise to the indicator function of $B^+_\rho(y)$ in $\mathbb{R}^{m+1}_+$. Furthermore,

$$\frac{\partial \eta_j}{\partial x_i}(x) = \begin{cases} -j\frac{x_i-y_i}{|x-y|}\chi'(j(\rho-|x-y|)) & \text{if } |x-y| \in \left(\rho-\frac{1}{j}, \rho-\frac{1}{2j}\right) \\ 0 & \text{if } |x-y| \geq \rho-\frac{1}{2j} \text{ or } |x-y| \leq \rho-\frac{1}{j}. \end{cases} \tag{3.3.5}$$

Replacing η with η_j in (3.3.4) we see that

$$\begin{aligned} &(m-1+\beta)\int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1}\left|\nabla v\right|^2 \eta_j \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} j|x-y|\chi'(j(\rho-|x-y|))\left|\nabla v\right|^2 \mathrm{d}x \\ &= -2\int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1}\frac{j}{|x-y|}\chi'(j(\rho-|x-y|))\left|(x-y)\cdot\nabla v\right|^2 \mathrm{d}x. \end{aligned} \tag{3.3.6}$$

We want to let $j \to \infty$ in (3.3.6) and so we consider each term in turn. Firstly, an application of Lebesgue's Dominated Convergence Theorem gives

$$\lim_{j\to\infty}\int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1}\left|\nabla v\right|^2 \eta_j \mathrm{d}x = \int_{B^+_\rho(y)} x^\beta_{m+1}\left|\nabla v\right|^2 \mathrm{d}x. \tag{3.3.7}$$

We must now deal with the terms in (3.3.6) involving χ'. Let

$$f(r) = \int_{\partial^+ B^+_r(y)} x^\beta_{m+1}\left|\nabla v\right|^2 \mathrm{d}\sigma(x)$$

for $r > 0$, where $\mathrm{d}\sigma$ is the measure on $\partial^+ B^+_r(y)$, and note that

$$\int_{B^+_\rho(y)} x^\beta_{m+1}\left|\nabla v\right|^2 \mathrm{d}x = \int_0^\rho f(r)\mathrm{d}r.$$

We calculate

$$\begin{aligned} \left|\int_{\rho-\frac{1}{j}}^{\rho} jr\chi'(j(\rho-r))f(r)\mathrm{d}r - \rho f(\rho)\right| &= \left|\int_{\rho-\frac{1}{j}}^{\rho} j\chi'(j(\rho-r))(rf(r)-\rho f(\rho))\mathrm{d}r\right| \\ &\leq cj\int_{\rho-\frac{1}{j}}^{\rho} |rf(r)-\rho f(\rho)|\mathrm{d}r. \end{aligned}$$

Moreover, for almost every $\rho > 0$, the right hand side tends to zero as $j \to \infty$ by Lebesgue's differentiation theorem. Hence, since $\chi'(j(\rho - r))$ is supported for

$r \in \left(\rho - \frac{1}{j}, \rho - \frac{1}{2j}\right)$, we find

$$\begin{aligned}
&\lim_{j\to\infty} \int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} j|x-y|\chi'(j(\rho - |x-y|)) \left|\nabla v\right|^2 \mathrm{d}x \\
&= \lim_{j\to\infty} \int_{\rho-\frac{1}{j}}^{\rho-\frac{1}{2j}} jr\chi'(j(\rho - r))f(r)\mathrm{d}r \\
&= \rho f(\rho) \\
&= \rho \int_{\partial^+ B^+_\rho(y)} x^\beta_{m+1} \left|\nabla v\right|^2 \mathrm{d}\sigma(x) \qquad (3.3.8)
\end{aligned}$$

for almost every $\rho > 0$. Similarly, Lebesgue's differentiation theorem guarantees that

$$\begin{aligned}
&\lim_{j\to\infty} \int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} \frac{j}{|x-y|}\chi'(j(\rho - |x-y|)) \left|(x-y)\cdot\nabla v\right|^2 \mathrm{d}x \\
&= \rho^{-1} \int_{\partial^+ B^+_\rho(y)} x^\beta_{m+1} \left|(x-y)\cdot\nabla v\right|^2 \mathrm{d}\sigma(x) \qquad (3.3.9)
\end{aligned}$$

for almost every $\rho > 0$. Hence, in view of (3.3.7), (3.3.8) and (3.3.9), letting $j \to \infty$ in (3.3.6) yields

$$\begin{aligned}
(m-1+\beta) \int_{B^+_\rho(y)} x^\beta_{m+1} \left|\nabla v\right|^2 \mathrm{d}x - \rho \int_{\partial B^+_\rho(y)} x^\beta_{m+1} \left|\nabla v\right|^2 \mathrm{d}\sigma(x) \\
= -\frac{2}{\rho} \int_{\partial^+ B^+_\rho(y)} x^\beta_{m+1} \left|(x-y)\cdot\nabla v\right|^2 \mathrm{d}\sigma(x).
\end{aligned}$$

Multiplying the above by the factor $-\rho^{-(\beta+m)}$ gives

$$\begin{aligned}
(1-m-\beta)\rho^{-(\beta+m)} \int_{B^+_\rho(y)} x^\beta_{m+1} \left|\nabla v\right|^2 \mathrm{d}x + \rho^{1-m-\beta} \int_{\partial^+ B^+_\rho(y)} x^\beta_{m+1} \left|\nabla v\right|^2 \mathrm{d}\sigma(x) \\
= 2\rho^{-(1+m+\beta)} \int_{\partial^+ B^+_\rho(y)} x^\beta_{m+1} \left|(x-y)\cdot\nabla v\right|^2 \mathrm{d}\sigma(x).
\end{aligned} \qquad (3.3.10)$$

Since $\frac{\mathrm{d}}{\mathrm{d}\rho} \int_{B^+_\rho(y)} x^\beta_{m+1} \left|\nabla v\right|^2 \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}\rho} \int_0^\rho f(r)\mathrm{d}r = \int_{\partial^+ B^+_\rho(y)} x^\beta_{m+1} \left|\nabla v\right|^2 \mathrm{d}\sigma(x)$ for almost all $\rho > 0$, in view of (3.3.10) we have

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^{1-m-\beta} \int_{B^+_\rho(y)} x^\beta_{m+1} \left|\nabla v\right|^2 \mathrm{d}x\right) = 2 \int_{\partial^+ B^+_\rho(y)} x^\beta_{m+1} \frac{\left|(x-y)\cdot\nabla v\right|^2}{|x-y|^{m+1+\beta}} \mathrm{d}\sigma(x) \qquad (3.3.11)$$

for almost every $\rho > 0$. Choose two positive numbers r and s with $0 < s < r$ such that $B_s^+(y) \subset B_r^+(y)$. Since the integral of an L^1 function over a half-ball of radius ρ is an absolutely continuous function of ρ we may integrate (3.3.11) between s and r. This gives (3.3.2) which in turn shows that $\rho^{1-m-\beta} \int_{B_\rho^+(y)} x_{m+1}^\beta \left|\nabla v\right|^2 \mathrm{d}x$ is a non-decreasing function of ρ. $\square$

It follows from Lemma 3.3.1.1 that $\lim_{\rho \to 0^+} \rho^{1-m-\beta} \int_{B_\rho^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x$ is well defined for y in $\mathcal{O}$ whenever v is a weakly stationary harmonic map with respect to the Neumann type boundary condition (3.1.11). This means we can define the *density function*

$$\Theta_v^\beta(y) = \lim_{\rho \to 0^+} \rho^{1-m-\beta} \int_{B_\rho^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x$$

analogously to Definition 1 in Section 2.5 of [46]. This function will be useful later in considerations of the singular set of minimisers and satisfies the following property.

Lemma 3.3.1.2. *The density function Θ_v^β is upper semi-continuous in $\mathcal{O}$ for any map v which is weakly stationary harmonic with respect to the Neumann-type boundary condition* (3.1.11)*.*

Proof. We need to show that if $(x_i)_{i \in \mathbb{N}}$ is a sequence in $\mathcal{O}$ converging to $x_0 \in \mathcal{O}$ then

$$\limsup_{i \to \infty} \Theta_v^\beta(x_i) \leq \Theta_v^\beta(x_0). \tag{3.3.12}$$

Consider such a sequence $(x_i)_{i \in \mathbb{N}}$ with $x_i \to x_0$ as $i \to \infty$ and let $\delta > 0$. We choose a radius $r > 0$ such that $\overline{\partial^0 B_r^+(x_0)} = \overline{B_r^m(x_0)} \subset \mathcal{O}$ and

$$r^{1-m-\beta} \int_{B_r^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \leq \Theta_v^\beta(x_0) + \delta.$$

For large i it follows that $x_i \in B_r^m(x_0)$ and so $B_{r-|x_i-x_0|}^+(x_i) \subset B_r^+(x_0)$. Thus

$$\begin{aligned}
&(r - |x_i - x_0|)^{1-m-\beta} \int_{B_{r-|x_i-x_0|}^+(x_i)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \\
&\leq \left(1 - \frac{|x_i - x_0|}{r}\right)^{1-m-\beta} r^{1-m-\beta} \int_{B_r^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \\
&\leq \left(1 - \frac{|x_i - x_0|}{r}\right)^{1-m-\beta} (\Theta_v^\beta(x_0) + \delta).
\end{aligned} \tag{3.3.13}$$

Using the definition of Θ_v^β and the boundary monotonicity formula, Lemma 3.3.1.1, we deduce that

$$\begin{aligned}\Theta_v^\beta(x_i) &= \lim_{\rho\to 0^+} \rho^{1-m-\beta} \int_{B_\rho^+(x_i)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \\ &\leq (r - |x_i - x_0|)^{1-m-\beta} \int_{B^+_{r-|x_i-x_0|}(x_i)} x_{m+1}^\beta |\nabla u|^2 \mathrm{d}x. \end{aligned} \tag{3.3.14}$$

The combination of (3.3.13) and (3.3.14) shows that

$$\Theta_v^\beta(x_i) \leq \left(1 - \frac{|x_i - x_0|}{r}\right)^{1-m-\beta} (\Theta_v^\beta(x_0) + \delta)$$

and therefore

$$\limsup_{i\to\infty} \Theta_v^\beta(x_i) \leq \limsup_{i\to\infty} \left(1 - \frac{|x_i - x_0|}{r}\right)^{1-m-\beta} (\Theta_v^\beta(x_0) + \delta) = \Theta_v^\beta(x_0) + \delta.$$

Since δ is arbitrary we conclude (3.3.12) holds as required. $\square$

Now we proceed with a derivation of the monotonicity formula for balls with closure in $\mathbb{R}^{m+1}_+$.

3.3.2 Interior Energy Monotonicity

The interior version of the energy monotonicity formula presented here is due to Grosse-Brauckmann, [22] Theorem 1. We present a proof here with a view to determining explicit dependences of the constants, this will allow a degree of compatibility of the boundary and interior versions of the lemma.

Lemma 3.3.2.1. *Suppose $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ is weakly stationary harmonic. Fix a ball $B_{\rho_0}(y)$ with $\overline{B_{\rho_0}(y)} \subset \mathbb{R}^{m+1}_+$ for some $\rho_0 > 0$. Choose two positive numbers r and s with $0 < s < r < \rho_0$. Then*

$$\begin{aligned} e^{rC|\beta|} r^{1-m} \int_{B_r(y)} x_{m+1}^\beta |\nabla v|^2 \,\mathrm{d}x \; - \; e^{sC|\beta|} s^{1-m} \int_{B_s(y)} x_{m+1}^\beta |\nabla v|^2 \,\mathrm{d}x \\ \geq 2 \int_{B_r(y)\setminus B_s(y)} x_{m+1}^\beta e^{|x-y|C|\beta|} \frac{|(x-y)\cdot \nabla v|^2}{|x-y|^{m+1}} \mathrm{d}x \end{aligned} \tag{3.3.15}$$

and therefore, for $0 < \rho < \rho_0$,

$$e^{\rho C|\beta|} \rho^{1-m} \int_{B_\rho(y)} x_{m+1}^\beta |\nabla v|^2 \,\mathrm{d}x$$

is a non-decreasing function of ρ where $C = (y_{m+1} - \rho_0)^{-1}$.

Proof. Again we follow [46] Section 2.4 and [33] Lemma 3.3, the method of proof is analogous to the proof of Lemma 3.3.1.1.

Fix ρ_0 and $y \in \mathbb{R}^{m+1}_+$ such that $\overline{B_{\rho_0}(y)} \subset \mathbb{R}^{m+1}_+$ and let $\eta \in C_0^\infty(\mathbb{R}^{m+1}_+)$. In order to proceed we test (3.1.16) against the function $(x-y)\eta(x)$. This gives

$$\begin{aligned}(m-1)&\int_{\mathbb{R}^{m+1}_+} x_{m+1}^\beta |\nabla v|^2 \eta \mathrm{d}x + \int_{\mathbb{R}^{m+1}_+} x_{m+1}^\beta (x-y)\cdot\nabla\eta\, |\nabla v|^2\, \mathrm{d}x \\ &+ \int_{\mathbb{R}^{m+1}_+} \beta x_{m+1}^\beta \eta |\nabla v|^2 \mathrm{d}x - \int_{\mathbb{R}^{m+1}_+} \beta x_{m+1}^{\beta-1} y_{m+1}\eta|\nabla v|^2 \mathrm{d}x \\ &= \int_{\mathbb{R}^{m+1}_+} x_{m+1}^\beta 2\left\langle (x-y)\cdot\nabla v, \nabla\eta\cdot\nabla v\right\rangle \mathrm{d}x. \end{aligned} \tag{3.3.16}$$

We consider a sequence of test functions which converge pointwise to the indicator function of $B_\rho(y)$, with $\rho < \rho_0$. Let $\chi \in C_0^\infty(\mathbb{R};[0,1])$ be the cutoff function given by (3.3.1). As in the proof of Lemma 3.3.1.1 we define $\eta_j(x) = \chi(j(\rho - |x-y|))$ and, replacing η with η_j in (3.3.16), we take the limit as $j \to \infty$ using Lebesgue's dominated convergence and differentiation theorems. We find

$$\begin{aligned}(m-1)&\int_{B_\rho(y)} x_{m+1}^\beta |\nabla v|^2\, \mathrm{d}x + \int_{B_\rho(y)} \beta x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \\ &- \int_{B_\rho(y)} \beta x_{m+1}^{\beta-1} y_{m+1} |\nabla v|^2 \mathrm{d}x - \rho\int_{\partial B_\rho(y)} x_{m+1}^\beta |\nabla v|^2\, \mathrm{d}\sigma(x) \\ &= -\frac{2}{\rho}\int_{\partial B_\rho(y)} x_{m+1}^\beta |(x-y)\cdot\nabla v|^2\, \mathrm{d}\sigma(x) \end{aligned} \tag{3.3.17}$$

where $\mathrm{d}\sigma(x)$ is the measure on $B_\rho(y)$. To proceed we must examine the term $\int_{B_\rho(y)} \beta x_{m+1}^{\beta-1} y_{m+1}|\nabla v|^2 \mathrm{d}x$ more closely. Observe that

$$x_{m+1} - \rho \leq y_{m+1} \leq x_{m+1} + \rho$$

and $(x_{m+1})^{-1} \leq (y_{m+1} - \rho_0)^{-1}$ for every $x \in B_\rho(y)$ and every $\rho < \rho_0$. We therefore calculate

$$\int_{B_\rho(y)} \beta x_{m+1}^{\beta-1} y_{m+1}|\nabla v|^2 \mathrm{d}x \leq \rho C|\beta| \int_{B_\rho(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x + \beta\int_{B_\rho(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \tag{3.3.18}$$

where $C = (y_{m+1} - \rho_0)^{-1}$. Substituting (3.3.18) into (3.3.17) we see that

$$-(1 - m + \rho C|\beta|) \int_{B_\rho(y)} x_{m+1}^\beta |\nabla v|^2 \,\mathrm{d}x - \rho \int_{\partial B_\rho(y)} x_{m+1}^\beta |\nabla v|^2 \,\mathrm{d}\sigma(x)$$
$$\leq -\frac{2}{\rho} \int_{\partial B_\rho(y)} x_{m+1}^\beta |(x-y) \cdot \nabla v|^2 \,\mathrm{d}\sigma(x)$$

and multiplying through by $-e^{\rho C|\beta|}\rho^{-m}$ gives

$$\begin{aligned}(1 - m + \rho C|\beta|)e^{\rho C|\beta|}\rho^{-m} &\int_{B_\rho(y)} x_{m+1}^\beta |\nabla v|^2 \,\mathrm{d}x \\ + e^{\rho C|\beta|}\rho^{1-m} &\int_{\partial B_\rho(y)} x_{m+1}^\beta |\nabla v|^2 \,\mathrm{d}\sigma(x) \\ \geq \rho^{-(1+m)} e^{\rho C|\beta|} 2 &\int_{\partial B_\rho(y)} x_{m+1}^\beta |(x-y) \cdot \nabla v|^2 \,\mathrm{d}\sigma(x).\end{aligned}$$

Thus for almost all admissible $\rho > 0$ we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\rho} \left(e^{\rho C|\beta|}\rho^{1-m} \int_{B_\rho(y)} x_{m+1}^\beta |\nabla v|^2 \,\mathrm{d}x \right) \\ &\geq 2 \int_{\partial B_\rho(y)} x_{m+1}^\beta e^{|x-y|C|\beta|} \frac{|(x-y) \cdot \nabla v|^2}{|x-y|^{(m+1)}} \mathrm{d}\sigma(x).\end{aligned} \tag{3.3.19}$$

Choose two positive numbers r and s with $0 < s < r < \rho_0$ such that $B_s(y) \subset B_r(y) \subset B_{\rho_0}(y)$. Since the integral of an L^1 function over a ball of radius ρ is an absolutely continuous function of ρ we may integrate (3.3.19) with respect to ρ. This gives (3.3.15) which in turn shows that $e^{\rho C|\beta|}\rho^{1-m} \int_{B_\rho(y)} x_{m+1}^\beta |\nabla v|^2 \,\mathrm{d}x$ is a non-decreasing function of ρ whenever $\rho \leq \rho_0$. □

Remark 3.3.2.1. Note that $y_{m+1} - \rho_0 = \mathrm{dist}(B_{\rho_0}(y), \mathbb{R}^m \times \{0\})$. The constant C from the lemma can thus be written as $C = (\mathrm{dist}(B_{\rho_0}(y), \mathbb{R}^m \times \{0\}))^{-1}$.

3.4 Motivating Observations Concerning the Energy

Here we consider the relationship between Dirichlet energies with respect to the Riemannian metric g defined by (3.0.1) and the Euclidean metric, taken over half balls $B_R^+(x_0) \subset \mathbb{R}_+^{m+1}$ with $x_0 \in \mathbb{R}^m \times \{0\}$ and balls $B_\rho(y)$ with $\overline{B_\rho(y)} \subset \mathbb{R}_+^{m+1}$ respectively. We also investigate the effect of changing between the Lebesgue measure and $\mathrm{d}\mu_\beta$ (corresponding to the metrics g for $\beta \in (-1, 1)$) for balls with closure contained in $\mathbb{R}^{m+1}_+$. The following discussion provides motivation for the form of Lemma 3.5.0.2 in Section 3.5 and our overall approach to proving the Hölder continuity of minimising harmonic maps.

First we consider switching between the Lebesgue measure and $\mathrm{d}\mu_\beta$, as we will need our conclusions in order to discuss the Dirichlet energies. We identify a class of ball in $\mathbb{R}^{m+1}_+$ on which we can immediately obtain results like the Poincaré inequality with respect to the measure $\mathrm{d}\mu_\beta$ from the corresponding results for the Lebesgue measure, recall Lemma 2.3.3.2 from Section 2.3.3, the only cost being a slightly larger constant. Let $B_\rho(y)$ be a ball with $\overline{B_\rho(y)} \subset \mathbb{R}^{m+1}_+$. We note that

$$\inf_{B_\rho(y)} x_{m+1}^\beta = (y_{m+1} - \mathrm{sgn}(\beta)\rho)^\beta$$

and

$$\sup_{B_\rho(y)} x_{m+1}^\beta = (y_{m+1} + \mathrm{sgn}(\beta)\rho)^\beta.$$

where $\mathrm{sgn}(\beta)$ is the sign of β. Let $f, g \in L^1(B_\rho(y); \mathbb{R}^n)$ and suppose

$$\int_{B_\rho(y)} |f| \mathrm{d}x \leq \int_{B_\rho(y)} |g| \mathrm{d}x.$$

We have

$$\begin{aligned}(y_{m+1} + \mathrm{sgn}(\beta)\rho)^{-\beta} \int_{B_\rho(y)} |f| \mathrm{d}\mu_\beta &\leq \int_{B_\rho(y)} |f| \mathrm{d}x \\ &\leq \int_{B_\rho(y)} |g| \mathrm{d}x \\ &\leq (y_{m+1} - \mathrm{sgn}(\beta)\rho)^{-\beta} \int_{B_\rho(y)} |g| \mathrm{d}\mu_\beta.\end{aligned}$$

Hence

$$\int_{B_\rho(y)} |f| \mathrm{d}\mu_\beta \leq \frac{(y_{m+1} + \mathrm{sgn}(\beta)\rho)^\beta}{(y_{m+1} - \mathrm{sgn}(\beta)\rho)^\beta} \int_{B_\rho(y)} |g| \mathrm{d}\mu_\beta$$

which is equivalent to

$$\int_{B_\rho(y)} |f| \mathrm{d}\mu_\beta \leq \left(1 + \frac{2\rho}{y_{m+1} - \rho}\right)^{|\beta|} \int_{B_\rho(y)} |g| \mathrm{d}\mu_\beta. \tag{3.4.1}$$

Note that $y_{m+1} - \rho = \mathrm{dist}(B_\rho(y), \mathbb{R}^m \times \{0\})$. In view of (3.4.1) we will restrict focus to balls where $\operatorname{dist}(B_\rho(y), \mathbb{R}^m \times \{0\}) \geq \rho$ since in this case (3.4.1) implies

$$\int_{B_\rho(y)} |f| \mathrm{d}\mu_\beta \leq 3^{|\beta|} \int_{B_\rho(z)} |g| \mathrm{d}\mu_\beta.$$

Furthermore, if $\beta \in (-1, 1)$ we have

$$\int_{B_\rho(y)} |f| \mathrm{d}\mu_\beta \leq 3 \int_{B_\rho(z)} |g| \mathrm{d}\mu_\beta. \tag{3.4.2}$$

In a similar vein we observe that, on any $B_\rho(y) \subset \mathbb{R}^{m+1}_+$ with $y_{m+1} - \rho \geq \rho$, we have $\frac{y_{m+1}}{2} \leq y_{m+1} - \rho \leq y_{m+1} \leq y_{m+1} + \rho \leq \frac{3y_{m+1}}{2}$. Thus if $x \in B_\rho(y)$ and $\beta \in (-1, 1)$ we have

$$c y_{m+1}^\beta \leq x_{m+1}^\beta \leq C y_{m+1}^\beta \tag{3.4.3}$$

for two constants c, C independent of β. It follows that

$$c_0 \leq \frac{\sup_{B_\rho(y)} x_{m+1}^\beta}{\inf_{B_\rho(y)} x_{m+1}^\beta} \leq C_0 \tag{3.4.4}$$

for two constants c_0, C_0 independent of $\beta \in (-1, 1)$.

We define some notation for the aforementioned class of ball. Let

$$\mathcal{B} = \{B_\rho(y) \subset \mathbb{R}^{m+1}_+ : y_{m+1} \geq 2\rho\}. \tag{3.4.5}$$

In addition, define

$$\mathcal{B}_\theta = \{B_\rho(y) \subset \mathbb{R}^{m+1}_+ : y_{m+1} \geq \theta\rho\} \tag{3.4.6}$$

for $\theta \geq 2$. Then $\mathcal{B}_\theta \subset \mathcal{B}$ and $\mathcal{B}_2 = \mathcal{B}$. Many of our calculations on balls in $\mathcal{B}_\theta$ will be with reference to a half-ball $B_R^+(x_0)$ with $x_0 \in \partial\mathbb{R}^{m+1}_+$ and so we also define

$$\mathcal{B}_\theta(x_0, R, r) = \{B_\rho(y) \subset B_R^+(x_0) : y_{m+1} \geq \theta\rho, y \in B_r^+(x_0)\}, \tag{3.4.7}$$

dropping the subscript θ in the case $\theta = 2$. We will also often want to refer to a class of half-balls related to $B_R^+(x_0)$. Let

$$\mathcal{B}^+(x_0, R, r) = \{B_\rho^+(y) \subset B_R^+(x_0) : y_{m+1} = 0, |x_0 - y| < r, \rho \leq r\}. \tag{3.4.8}$$

For future reference we note that the classes defined in (3.4.8) and (3.4.7) correspond to those considered in (3.5.3) and (3.5.4) respectively in Lemma 3.5.0.2.

When combined with, for example, the Poincaré inequality, Lemma 2.3.3.2, for functions in $W^{1,2}(B_\rho(y); \mathbb{R}^n)$ where $B_\rho(y) \in \mathcal{B}$, (3.4.2) and (3.4.4) are particularly useful. We record the following version of Lemma 2.3.3.2.

Lemma 3.4.0.1. *Let $B_\rho(y) \in \mathcal{B}$ and $v \in W^{1,2}_\beta(B_\rho(y);\mathbb{R}^n)$. Then*

$$\int_{B_\rho(y)} |v - \overline{v}_{B_\rho(y),\beta}|^2 \mathrm{d}\mu_\beta \leq C\rho^2 \int_{B_\rho(y)} |\nabla v|^2 \mathrm{d}\mu_\beta \tag{3.4.9}$$

for a positive constant $C = C(m)$.

Proof. For such $B_\rho(y)$, we combine Lemma 2.3.3.2 with (3.4.4) to yield the lemma. ☐

Now we discuss the Dirichlet energies, taken over open $\Omega \subset \mathbb{R}^{m+1}_+$. For reference, we define the energies

$$E^\beta_\Omega(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \,\mathrm{d}\mu_\beta \tag{3.4.10}$$

for $v \in W^{1,2}_\beta(\Omega;\mathbb{R}^n)$ and

$$E_\Omega(v) = E^0_\Omega(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \,\mathrm{d}x$$

for $v \in W^{1,2}(\Omega;\mathbb{R}^n)$. We will favour the integral notation in technical computations involving the weights to make them more transparent. When we change between $\mathrm{d}\mu_\beta$ and the Lebesgue measure, or want to emphasise the role x^β_{m+1} has as a coefficient of a PDE, in addition to a weighting of the Lebesgue measure, we may expand the expression for $\mathrm{d}\mu_\beta$ into $x^\beta_{m+1}\mathrm{d}x$.

Consider $B_{\frac{y_{m+1}}{2}}(y) \in \mathcal{B}$, let $y^+ = (y_1,\dots,y_m,0)$ and note that $B_{\frac{y_{m+1}}{2}}(y) \subset B^+_{\frac{3y_{m+1}}{2}}(y^+)$. Using (3.4.3) we have

$$\begin{aligned}\left(\frac{y_{m+1}}{2}\right)^{1-m}\int_{B_{\frac{y_{m+1}}{2}}(y)} |\nabla v|^2 \,\mathrm{d}x &\leq C y^{-\beta}_{m+1}\left(\frac{y_{m+1}}{2}\right)^{1-m}\int_{B_{\frac{y_{m+1}}{2}}(y)} x^\beta_{m+1}|\nabla v|^2 \,\mathrm{d}x \\ &\leq C\left(\frac{3y_{m+1}}{2}\right)^{1-m-\beta}\int_{B^+_{\frac{3y_{m+1}}{2}}(y^+)} x^\beta_{m+1}|\nabla v|^2 \,\mathrm{d}x\end{aligned} \tag{3.4.11}$$

where $C = C(m)$ is a positive constant which is chosen large enough to be independent of $\beta \in (-1,1)$. In fact, we can combine (3.4.11) with the monotonicity formulas proved in Section 3.3 and replace the left hand side of (3.4.11) with $\rho^{1-m}\int_{B_\rho(y)} |\nabla v|^2 \,\mathrm{d}x$ whenever $B_\rho(y) \in \mathcal{B}$. Using this idea we can connect estimates for the energy on balls in $\mathbb{R}^{m+1}_+$ with estimates for the energy on half-balls centred on $\mathcal{O}$. More precisely, we have the following.

Lemma 3.4.0.2. *Suppose $\beta \in (-1,1)$ and let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ be a weakly stationary harmonic map with respect to the Neumann type boundary condition (3.1.11). Let $B^+_R(x_0)$ be a half-ball with $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$ and suppose $B_\rho(y) \in \mathcal{B}(x_0, R, \frac{R}{3})$. Then there is a constant $C = C(m)$ such that*

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \mathrm{d}x \leq C R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x. \tag{3.4.12}$$

Proof. To prove the lemma we need to combine the boundary energy monotonicity formula, Lemma 3.3.1.1, with the interior energy monotonicity formula, Lemma 3.3.2.1, in a suitable manner. It follows from (3.4.3) that

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \,\mathrm{d}x \leq C y^{-\beta}_{m+1} \rho^{1-m} \int_{B_\rho(y)} x^\beta_{m+1} |\nabla v|^2 \,\mathrm{d}x \tag{3.4.13}$$

for every $B_\rho(y) \in \mathcal{B}$ and a constant C independent of $\beta \in (-1,1)$. Notice that any ball $B_\rho(y) \in \mathcal{B}$ satisfies $B_\rho(y) \subset B_{\frac{y_{m+1}}{2}}(y)$ so we can choose the scaling factor $e^{\frac{2|\beta|\rho}{y_{m+1}}}$ in Lemma 3.3.2.1. Furthermore, $e^{\frac{2|\beta|\rho}{y_{m+1}}} \leq e^{|\beta|} \leq e$ since $y_{m+1} \geq 2\rho$ and $\beta \in (-1,1)$. Hence, applying Lemma 3.3.2.1, we find

$$\begin{aligned}
& y^{-\beta}_{m+1} \rho^{1-m} \int_{B_\rho(y)} x^\beta_{m+1} |\nabla v|^2 \,\mathrm{d}x \\
& \leq y^{-\beta}_{m+1} e^{\frac{2|\beta|\rho}{y_{m+1}}} \rho^{1-m} \int_{B_\rho(y)} x^\beta_{m+1} |\nabla v|^2 \,\mathrm{d}x \\
& \leq y^{-\beta}_{m+1} e^{\frac{2|\beta| y_{m+1}}{2 y_{m+1}}} \left(\frac{y_{m+1}}{2}\right)^{1-m} \int_{B_{\frac{y_{m+1}}{2}}(y)} x^\beta_{m+1} |\nabla v|^2 \,\mathrm{d}x \\
& \leq 2^\beta e \left(\frac{y_{m+1}}{2}\right)^{1-m-\beta} \int_{B_{\frac{y_{m+1}}{2}}(y)} x^\beta_{m+1} |\nabla v|^2 \,\mathrm{d}x \\
& \leq C \left(\frac{y_{m+1}}{2}\right)^{1-m-\beta} \int_{B_{\frac{y_{m+1}}{2}}(y)} x^\beta_{m+1} |\nabla v|^2 \,\mathrm{d}x
\end{aligned} \tag{3.4.14}$$

where C is chosen large enough to be independent of $\beta \in (-1,1)$. Now let $y = (y_1, \ldots, y_{m+1})$ and $y^+ = (y_1, \ldots, y_m, 0)$. Since $B_{\frac{y_{m+1}}{2}}(y) \subset B^+_{\frac{3y_{m+1}}{2}}(y^+)$, we have

$$\left(\frac{y_{m+1}}{2}\right)^{1-m-\beta}\int_{B_{\frac{y_{m+1}}{2}}(y)} x_{m+1}^{\beta}\,|\nabla v|^2\,\mathrm{d}x$$

$$\leq C\left(\frac{3y_{m+1}}{2}\right)^{1-m-\beta}\int_{B^+_{\frac{3y_{m+1}}{2}}(y^+)} x_{m+1}^{\beta}\,|\nabla v|^2\,\mathrm{d}x \tag{3.4.15}$$

where $C = C(m)$ is chosen such that $C(m) \geq 3^m \geq 3^{m+\beta-1}$. If $B_\rho(y) \in \mathcal{B}\left(x_0, R, \frac{R}{3}\right)$, then $y \in B^+_{\frac{R}{3}}(x_0)$ and $B^+_{\frac{3y_{m+1}}{2}}(y^+) \subset B^+_{\frac{R}{2}}(y^+) \subset B^+_R(x_0)$. It follows from Lemma 3.3.1.1 that

$$\left(\frac{3y_{m+1}}{2}\right)^{1-m-\beta}\int_{B^+_{\frac{3y_{m+1}}{2}}(y^+)} x_{m+1}^{\beta}\,|\nabla v|^2\,\mathrm{d}x \leq \left(\frac{R}{2}\right)^{1-m-\beta}\int_{B^+_{\frac{R}{2}}(y^+)} x_{m+1}^{\beta}\,|\nabla v|^2\,\mathrm{d}x. \tag{3.4.16}$$

Lastly, since $B^+_{\frac{R}{2}}(y^+) \subset B^+_R(x_0)$ for $y \in B^+_{\frac{R}{2}}(x_0)$, it follows that

$$\left(\frac{R}{2}\right)^{1-m-\beta}\int_{B^+_{\frac{R}{2}}(y^+)} x_{m+1}^{\beta}\,|\nabla v|^2\,\mathrm{d}x \leq CR^{1-m-\beta}\int_{B^+_R(x_0)} x_{m+1}^{\beta}\,|\nabla v|^2\,\mathrm{d}x \tag{3.4.17}$$

where $C = C(m) \geq 2^m \geq 2^{m+\beta-1}$. A combination of (3.4.13), (3.4.14), (3.4.15), (3.4.16) and (3.4.17) yields (3.4.12). $\square$

A consequence of this lemma, vital to our subsequent regularity theory, is as follows. Suppose $R^{1-m-\beta}\int_{B^+_R(x_0)} x_{m+1}^{\beta}\,|\nabla v|^2\,\mathrm{d}x \leq \varepsilon$ for some $\varepsilon > 0$ and a half-ball $B^+_R(x_0)$ with $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$. The lemma states that we can control the energy $\rho^{1-m}\int_{B_\rho(y)}|\nabla v|^2\mathrm{d}x$, for all $B_\rho(y) \in \mathcal{B}(x_0, R, \frac{R}{3})$, in terms of ε. In particular, if ε is small then so is $\rho^{1-m}\int_{B_\rho(y)}|\nabla v|^2\mathrm{d}x$. analogously, Lemma 3.3.1.1 provides control of the energy $\rho^{1-m-\beta}\int_{B^+_\rho(y)} x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x$ in terms of ε for all half-balls $B^+_\rho(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$. The details of this process are given in the proof of Theorem 3.12.1.1.

The preceding two observations motivate the form of the decay lemma in the next section.

Lastly we explain an assumption we will frequently make in our lemmata; we assume that any half-ball $B^+_R(x_0)$ appearing henceforth has radius no greater than 1 and, in general, we will work in the region $\mathbb{R}^m \times (0, 1]$. This is a technical assumption based on the fact that for $x_{m+1} \in (0, 1]$ the weights x_{m+1}^{β} are ordered in the following sense; if $\beta_2 < \beta_1$ are real numbers in $(-1, 1)$ then $x_{m+1}^{\beta_1} \leq x_{m+1}^{\beta_2}$

whenever $x_{m+1} \in (0,1]$. If we assume $x_{m+1} \geq 1$ then the converse is true and the assumption $x_{m+1} \leq 1$ eliminates the need to consider more than one kind of ordering of the weights. This assumption has no impact on our regularity theory; the assumption of our main theorem will be

$$R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta \left|\nabla v\right|^2 \mathrm{d}x \leq \varepsilon$$

for a half-ball $B_R^+(x_0)$ with $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$ and an $\varepsilon > 0$ and in view of the monotonicity formula, Lemma 3.3.1.1,

$$r^{1-m-\beta} \int_{B_r^+(x_0)} x_{m+1}^\beta \left|\nabla v\right|^2 \mathrm{d}x \leq R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta \left|\nabla v\right|^2 \mathrm{d}x \leq \varepsilon$$

for every $r \leq R$. Thus we may always assume $R \leq 1$.

3.5 A Modified Lemma of Morrey

Since we are considering systems of partial differential equations derived from energy functionals it is appropriate to try to find an integral characterisation of Hölder continuity in terms of the energy. We modify a decay lemma of Morrey, [32] Theorem 3.5.2, using the version in [33] Lemma 2.1 for the proof. The form of the following lemma was suggested to us by the considerations in Section 3.4.

Recall the notation $\mathcal{B}_\theta(x_0, R, r)$ and $\mathcal{B}^+(x_0, R, r)$ defined in Section 3.4. We will also consider the averages $\overline{v}_\Omega = \frac{1}{\int_\Omega \mathrm{d}x} \int_\Omega v \mathrm{d}x$ and $\overline{v}_{\Omega,\beta} = \frac{1}{\int_\Omega \mathrm{d}\mu_\beta} \int_\Omega v \mathrm{d}\mu_\beta$ for bounded $\Omega \subset \mathbb{R}_+^{m+1}$ and correspondingly integrable v.

We will need the following version of the original decay lemma for the proof of the modified version.

Lemma 3.5.0.1 ([33] Lemma 2.1). *Let $\gamma > 0$, $v \in W^{1,p}(B_R(x_0); \mathbb{R}^n)$ for $p \geq 1$ and $a > 0$. There exists a constant $C_0 = C_0(m, \gamma)$ such that the following holds. Suppose*

$$\rho^{p-(1+m)-\gamma} \int_{B_\rho(y)} |\nabla v|^p \mathrm{d}x \leq a \tag{3.5.1}$$

for every $y \in B_{\frac{R}{2}}(x_0)$ and every $\rho \leq \frac{R}{2}$. Then, for almost every $x_1, x_2 \in B_{\frac{R}{2}}(x_0)$,

$$|v(x_1) - v(x_2)| \leq C_0 a^{\frac{1}{p}} |x_1 - x_2|^{\frac{\gamma}{p}}. \tag{3.5.2}$$

The modified statement is as follows.

Lemma 3.5.0.2. *Let $\beta \in (-1,1)$, $\gamma > 0$, $x_0 \in \mathbb{R}^m \times \{0\}$, $a > 0$, $\theta_1 \geq 2$ and $\theta_2 \leq \frac{1}{2}$. Define $\theta = \frac{\theta_2}{2\theta_1}$. Then there exists a constant $C_0 = C_0(m, \gamma, \theta_1, \beta)$ such that if $v \in W^{1,2}_\beta(B^+_R(x_0); \mathbb{R}^n)$ with*

$$r^{1-m-\beta-\gamma} \int_{B^+_r(x_1)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \leq a \tag{3.5.3}$$

for every $B^+_r(x_1) \in \mathcal{B}^+(x_0, R, \theta_2 R)$, and

$$r^{1-m-\gamma} \int_{B_r(x_2)} |\nabla v|^2 \mathrm{d}x \leq a \tag{3.5.4}$$

for every $B_r(x_2) \in \mathcal{B}_{\theta_1}(x_0, R, \theta_2 R)$, then for almost every $x_1, x_2 \in B^+_{\theta R}(x_0)$,

$$|v(x_1) - v(x_2)| \leq C_0 a^{\frac{1}{2}} |x_1 - x_2|^{\frac{\gamma}{2}}. \tag{3.5.5}$$

Proof. The underlying idea of the proof is to apply Lemma 3.5.0.1 for $p = 1$. In order to apply this lemma we consider the even reflection of v, with respect to $\partial\mathbb{R}^{m+1}_+$, to $B_R(x_0)$. Due to the nature of this reflection we only need to verify the assumptions of Lemma 3.5.0.1 hold for balls with centres y satisfying $y_{m+1} \geq 0$. We conclude the proof by using the assumptions (3.5.3) and (3.5.4) combined with applications of Hölder's inequality to verify that (3.5.1) is satisfied for all $B_r(x_1) \subset B_{2\theta R}(x_0)$ with $x_1 \in B_{\theta R}(x_0)$ and $r \leq \theta R$.

Recall that, by Lemma 2.4.2.1, the even reflection of v in $\partial\mathbb{R}^{m+1}_+$, which we do not relabel, is in $W^{1,2}_\beta(B_R(x_0); \mathbb{R}^n)$. First we show that we may restrict our attention to the case of balls with centre y satisfying $y_{m+1} \geq 0$. Suppose that the assumption (3.5.1) from Lemma 3.5.0.1 is satisfied, with $p = 1$, on every $B_\rho(y) \subset B_{2\theta R}(x_0)$ with $y_{m+1} \geq 0$, $y \in B_{\theta R}(x_0)$ and $\rho \leq \theta R$. This means

$$\rho^{-m} \int_{B_\rho(y)} |\nabla v| \mathrm{d}x \leq \tilde{a} \rho^{\tilde{\gamma}} \tag{3.5.6}$$

for an $\tilde{a} > 0$ and a $\tilde{\gamma} \in (0,1)$. Consider $B_\rho(z) \subset B_{2\theta R}(x_0)$ with $z_{m+1} \leq 0$, $z \in B_{\theta R}(x_0)$ and $\rho \leq \theta R$. It follows that $z = (y', -y_{m+1})$ for some $y = (y', y_{m+1})$ with $y_{m+1} \geq 0$ and $y \in B_{\theta R}(x_0)$. A change of variables gives

$$\int_{B_\rho(z)} |\nabla v| \mathrm{d}x = \int_{B_\rho(y)} |\nabla v| \mathrm{d}x. \tag{3.5.7}$$

It follows from (3.5.7) that (3.5.6) is satisfied on every $B_\rho(z) \subset B_{2\theta R}(x_0)$ with $z_{m+1} \leq 0$, $z \in B_{\theta R}(x_0)$ and $\rho \leq \theta R$ provided that (3.5.6) is satisfied on every

$B_\rho(y) \subset B_{2\theta R}(x_0)$ with $y_{m+1} \geq 0$, $y \in B_{\theta R}(x_0)$ and $\rho \leq \theta R$.

Our task now is to show that the assumptions of Lemma 3.5.0.1 hold on $B_{2\theta R}(x_0)$ for θ as specified. In particular, we show that (3.5.6) holds, for a $\tilde{\gamma} \in (0,1)$ and an $\tilde{a} > 0$, on every ball $B_r(x_1) \subset B_{2\theta R}(x_0)$ with $(x_1)_{m+1} \geq 0$, $x_1 \in B_{\theta R}(x_0)$ and $r \leq \theta R$. Such a ball must satisfy either $B_r(x_1) \in \mathcal{B}_{\theta_1}(x_0, 2\theta R, \theta R)$ or $B_r(x_1) \notin \mathcal{B}_{\theta_1}(x_0, 2\theta R, \theta R)$. We address these cases in turn. Henceforth, C will denote a constant depending only on m and we will not distinguish such C unless necessary.

Suppose that $B_r(x_1) \in \mathcal{B}_{\theta_1}(x_0, 2\theta R, \theta R)$ with $r \leq \theta R$. We apply Hölder's inequality to see that

$$\begin{aligned}
\int_{B_r(x_1)} |\nabla v| \mathrm{d}x &\leq \left(\int_{B_r(x_1)} \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{B_r(x_1)} |\nabla v|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&= C r^{\frac{m+1}{2}} \left(\int_{B_r(x_1)} |\nabla v|^2 \mathrm{d}x \right)^{\frac{1}{2}}. \qquad (3.5.8)
\end{aligned}$$

Now notice that $r^{\frac{m+1}{2}} = r^m r^{\frac{1-m}{2}}$. Furthermore, as $\theta \leq \frac{\theta_2}{2\theta_1} \leq \theta_2$, the assumption $B_r(x_1) \in \mathcal{B}_{\theta_1}(x_0, 2\theta R, \theta R)$ implies $B_r(x_1) \in \mathcal{B}_{\theta_1}(x_0, R, \theta_2 R)$. Thus we combine (3.5.8) with the hypothesis (3.5.4) to give

$$\begin{aligned}
r^{-m} \int_{B_r(x_1)} |\nabla v| \mathrm{d}x &\leq r^{-m} C r^m r^{\frac{1-m}{2}} \left(\int_{B_r(x_1)} |\nabla v|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&= C \left(r^{1-m} \int_{B_r(x_1)} |\nabla v|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq C a^{\frac{1}{2}} r^{\frac{\gamma}{2}} \qquad (3.5.9)
\end{aligned}$$

which is (3.5.6) with $\tilde{\gamma} = \frac{\gamma}{2}$ and $\tilde{a} = C a^{\frac{1}{2}}$.

We must now consider the case when $B_r(x_1) \subset B_{2\theta R}(x_0)$ with $x_1 \in B_{\theta R}(x_0)$, $(x_1)_{m+1} \geq 0$ and $r \leq \theta R$ but $B_r(x_1) \notin \mathcal{B}_{\theta_1}(x_0, 2\theta R, \theta R)$. In this case, since $B_r(x_1) \subset B_{2\theta R}(x_0)$ and $x_1 \in B_{\theta R}(x_0)$ by assumption, we must have $(x_1)_{m+1} < \theta_1 r$. Expressed slightly differently, we find $(x_1)_{m+1} - \zeta r < r$ where $\zeta \geq 1$ is such that $\theta_1 = \zeta + 1$. As a consequence, $B_r(x_1) \subset B_{2\zeta r + r}(x_1^+)$ where $x_1^+ = x_1 - (0, (x_1)_{m+1})$. Using the fact that v is even with respect to $\partial \mathbb{R}^{m+1}_+$ and applying Hölder's inequality we find

$$\begin{aligned}\int_{B_r(x_1)} |\nabla v| \mathrm{d}x &\leq \int_{B_{2\zeta r+r}(x_1^+)} |\nabla v| \mathrm{d}x \\ &= 2\int_{B^+_{2\zeta r+r}(x_1^+)} |\nabla v| \mathrm{d}x \\ &\leq 2\left(\int_{B^+_{2\zeta r+r}(x_1^+)} x_{m+1}^{-\beta} \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B^+_{2\zeta r+r}(x_1^+)} x_{m+1}^{\beta} |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}}.\end{aligned} \tag{3.5.10}$$

We bound $\left(\int_{B^+_{2\zeta r+r}(x_1^+)} x_{m+1}^{-\beta} \mathrm{d}x\right)^{\frac{1}{2}}$ above as follows. Let $Q_\rho(y)$ denote the cube, with edges parallel to the coordinate axis and of length 2ρ, centred at $y \in \partial\mathbb{R}^{m+1}_+$ and let $Q^+_\rho(y) = Q_\rho(y) \cap \mathbb{R}^{m+1}_+$. We calculate

$$\int_{B^+_{2\zeta r+r}(x_1^+)} x_{m+1}^{-\beta} \mathrm{d}x \leq \int_{Q^+_{2\zeta r+r}(x_1^+)} x_{m+1}^{-\beta} \mathrm{d}x = \frac{C}{1-\beta}(2\zeta r + r)^{1+m-\beta}. \tag{3.5.11}$$

Combining (3.5.11) with (3.5.10) gives

$$\int_{B_r(x_1)} |\nabla v| \mathrm{d}x \leq \frac{C}{(1-\beta)^{\frac{1}{2}}} (2\zeta r + r)^{\frac{1+m-\beta}{2}} \left(\int_{B^+_{2\zeta r+r}(x_1^+)} x_{m+1}^{\beta} |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}}. \tag{3.5.12}$$

We write $(2\zeta r + r)^{\frac{1+m-\beta}{2}} = (2\zeta+1)^m r^m (2\zeta r + r)^{\frac{1-m-\beta}{2}}$ and use (3.5.12) to deduce that

$$\begin{aligned}&r^{-m}\int_{B_r(x_1)} |\nabla v| \mathrm{d}x \\ &\leq r^{-m}\frac{C(2\zeta+1)^m}{(1-\beta)^{\frac{1}{2}}} r^m (2\zeta r + r)^{\frac{1-m-\beta}{2}} \left(\int_{B^+_{2\zeta r+r}(x_1^+)} x_{m+1}^{\beta} |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ &= \frac{C(2\zeta+1)^m}{(1-\beta)^{\frac{1}{2}}} \left((2\zeta r + r)^{1-m-\beta} \int_{B^+_{2\zeta r+r}(x_1^+)} x_{m+1}^{\beta} |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}}.\end{aligned} \tag{3.5.13}$$

In order to apply the assumption (3.5.3) on $B^+_{2\zeta r+r}(x_1^+)$ we must check that $B^+_{2\zeta r+r}(x_1^+) \in \mathcal{B}^+(x_0, R, \theta_2 R)$. Recall the assumptions $\theta \leq \frac{\theta_2}{2\theta_1} \leq \theta_2 \leq \frac{1}{2}$, $x_1 \in B_{\theta R}(x_0)$ and $r \leq \theta R$. It follows that $2\zeta r + r \leq 2\theta_1 r \leq \theta_2 R$ and $x_1^+ \in B^+_{\theta_2 R}(x_0)$ which, in turn, implies that $B^+_{2\zeta r+r}(x_1^+) \subset B^+_{\theta_2 R}(x_1^+) \subset B^+_{2\theta_2 R}(x_0) \subset B^+_R(x_0)$ since

$\theta_2 \leq \frac{1}{2}$. Therefore, we may apply (3.5.3) and thus, in view of (3.5.13),

$$r^{-m} \int_{B_r(x_1)} |\nabla v| \mathrm{d}x \leq \frac{C(2\zeta+1)^{m+\frac{\gamma}{2}}}{(1-\beta)^{\frac{1}{2}}} a^{\frac{1}{2}} r^{\frac{\gamma}{2}} \tag{3.5.14}$$

which is (3.5.6) with $\tilde{\gamma} = \frac{\gamma}{2}$ and $\tilde{a} = \frac{C(2\zeta+1)^{m+\frac{\gamma}{2}}}{(1-\beta)^{\frac{1}{2}}} a^{\frac{1}{2}}$. Recalling (3.5.7), we combine (3.5.9) and (3.5.14) to see that (3.5.6) is satisfied for every $B_r(x_1) \subset B_{2\theta R}(x_0)$ with $x_1 \in B_{\theta R}(x_0)$ and $r \leq \theta R$. Thus we may apply Lemma 3.5.0.1 to conclude the proof. □

3.5.1 From Decay Estimates to Hölder continuity

We will require a method to pass from estimates which constitute improvements of the monotonicity formulas given by lemmata 3.3.1.1 and 3.3.2.1, to estimates of the form (3.5.3) and (3.5.4). Proving such estimates is our primary concern hereafter. In particular, we will show that on any $B_R^+(x_0)$ with $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$, provided $R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^{\beta} |\nabla v|^2 \mathrm{d}x$ is sufficiently small, the following holds; for $\theta \geq 2$ and $\sigma_1, \sigma_2 \in (0,1)$ to be determined, if $B_r(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$ and $\rho \leq r$ we have

$$e^{\frac{|\beta|\sigma_1\rho}{y_{m+1}-\rho}} (\sigma_1\rho)^{1-m} \int_{B_{\sigma\rho}(y)} |\nabla v|^2 \mathrm{d}x \leq \frac{1}{2} e^{\frac{|\beta|\rho}{y_{m+1}-\rho}} \rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \mathrm{d}x \tag{3.5.15}$$

and if $B_\rho^+(y) \in \mathcal{B}^+(x_0, R, \frac{R}{3})$ and $r \leq \rho$ we have

$$(\sigma_2\rho)^{1-m-\beta} \int_{B_{\sigma_2\rho}^+(y)} |\nabla v|^2 \mathrm{d}x \leq \frac{1}{2} \rho^{1-m-\beta} \int_{B_\rho^+(y)} |\nabla v|^2 \mathrm{d}x. \tag{3.5.16}$$

Once we know either of these estimates holds we can apply the following lemma, possibly combined with a few additional arguments, to deduce that the corresponding hypothesis in Lemma 3.5.0.2 holds.

Lemma 3.5.1.1 (Lemma 8.23 in [21]). *Let $f, h : (0, R_0] \to \mathbb{R}$ be non-decreasing functions which satisfy*

$$f(\sigma r) \leq \tau f(r) + h(r) \tag{3.5.17}$$

for every $r \leq R_0$ and some fixed $\sigma, \tau \in (0,1)$. There exists a $C = C(\sigma,\tau) > 0$ and, for every $\mu \in (0,1)$, there exists a $\gamma = \gamma(\sigma,\tau,\mu) > 0$ such that

$$f(r) \leq C \left(\frac{r}{R_0}\right)^{\gamma} f(R_0) + h(r^{\mu} R_0^{1-\mu}). \tag{3.5.18}$$

The rest of this section is dedicated to showing minimisers of E^β satisfy (3.5.15) and (3.5.16) in any $B_R^+(x_0)$ with $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$ whenever the energy $R^{1-m-\beta}\int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x$ is small.

3.6 Interior Estimates for Hölder continuity

Here we show, using the regularity theory of Schoen and Uhlenbeck [44] and Schoen [45], that minimisers $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ of E^β relative to $\mathcal{O}$ satisfy (3.5.15) from the preceeding section, and consequently (3.5.4) in Lemma 3.5.0.2, provided the scaled energy $R^{1-m-\beta}\int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x$ is sufficiently small. In other words, we show that, given a half-ball $B_R^+(x_0)$ with $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$ and a minimiser v with $R^{1-m-\beta}\int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x$ small, if $B_r(y) \in \mathcal{B}_\theta\left(x_0, R, \frac{R}{3}\right)$ for a $\theta \geq 2$ to be chosen, then we have

$$\rho^{1-m}\int_{B_\rho(y)} |\nabla v|^2 \,\mathrm{d}x \leq C\left(\frac{\rho}{r}\right)^\gamma r^{1-m}\int_{B_r(y)} |\nabla v|^2 \,\mathrm{d}x \tag{3.6.1}$$

for some $\gamma \in (0,1)$ and any $\rho \leq r$. We recall the relevant theory from [44] and [45] sections 1, 2 and 3, stating the results in our context with slightly different notation.

Let $B_r(y)$ be the ball of radius $r > 0$ in $\mathbb{R}^{m+1}$ centred at y. The Sobolev spaces, considered in [44] and [45], that we need here are subsets of $W^{1,2}(\overline{B_r(y)}; \mathbb{R}^n)$, defined for maps with compact domain $\overline{B_r(y)}$. It follows from theorem 3.18 in [1] that $W^{1,2}(\overline{B_r(y)}; \mathbb{R}^n)$ coincides with the space $W^{1,2}(B_r(y); \mathbb{R}^n)$. Thus, as in [44] and [45], we consider maps in the Sobolev spaces

$$W^{1,2}(B_r(y); N) = \{v \in W^{1,2}(B_r(y); \mathbb{R}^n) : v(x) \in N \text{ for almost every } x \in B_r(y)\}.$$

In order to prove (3.6.1) holds with a constant independent of the balls $B_r(y) \in \mathcal{B}_\theta\left(x_0, R, \frac{R}{3}\right)$, we take advantage of the fact that on each $B_r(y) \in \mathcal{B}$ we may multiply the metric $g(x)$, defined in (3.0.1), whose components are $g_{ij}(x) = x_{m+1}^\alpha \delta_{ij}$, by a constant factor, namely $y_{m+1}^{-\alpha}$, to get a metric which is close to the Euclidean metric in a sense which uniform across $B_r(y) \in \mathcal{B}_\theta\left(x_0, R, \frac{R}{3}\right)$. The energy functionals corresponding to these metrics on a given $B_r(y)$ are multiples of the energies defined by (3.4.10) in Section 3.4. In particular, they have the form

$$2y_{m+1}^{-\beta} E_{B_r(y)}^\beta(v) = \int_{B_r(y)} (y_{m+1}^{-1} x_{m+1})^\beta |\nabla v|^2 \mathrm{d}x$$

for $v \in W^{1,2}(B_r(y); N)$. Critical points of this energy with respect to variations of the independent variable are weakly harmonic maps as defined in Chapter 1. From this definition we see that they satisfy (3.1.6), multiplied on both sides by the factor $2y_{m+1}^{-\beta}$, but with $\phi \in C_0^\infty(B_r(y); \mathbb{R}^n)$. In other words, a map $v \in W^{1,2}(B_r(y); N)$ is weakly harmonic with respect to the metric $y_{m+1}^{-\alpha} g$ on $B_r(y)$ if

$$\int_{B_r(y)} x_{m+1}^{\beta} \langle \nabla v, \nabla \phi \rangle \, \mathrm{d}x = \int_{B_r(y)} x_{m+1}^{\beta} \langle \phi, A(v)(\nabla v, \nabla v) \rangle \, \mathrm{d}x \tag{3.6.2}$$

for every $\phi \in C_0^\infty(B_r(y); \mathbb{R}^n)$, where A is the second fundamental form of N.

We now discuss minimisers as defined in [44] section 1. Consider the compact Riemannian manifold $\overline{B_1(0)}$ with metric $\tilde{g}$. Recall the following notation. The matrix representing $\tilde{g}$ is $\tilde{g}_{ij}$ for $i, j = 1, \ldots, m+1$, which has determinant $\det(\tilde{g})$. The inverse matrix then has components denoted $\tilde{g}^{ij}$ for $i, j = 1, \ldots, m+1$. Furthermore, we write $|\nabla v|_{\tilde{g}} = \sum_{i,j=1}^{m+1} \tilde{g}^{ij} \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_j} \right\rangle$. For each $\tilde{g}$, the energy functional on $\overline{B_1(0)}$ is given by

$$E_{\tilde{g}}(v) = \int_{B_1(0)} |\nabla v|_{\tilde{g}}^2 \sqrt{\det(\tilde{g})} \mathrm{d}x$$

where $v \in W^{1,2}(B_1(0); N)$. The definition of energy minimising corresponding to $E_{\tilde{g}}$ is as follows.

Definition 3.6.0.1. [[44] Section 1] Any $v \in W^{1,2}(B_1(0); N)$ is an $E_{\tilde{g}}$ *minimising map* if it satisfies $E_{\tilde{g}}(v) \leq E_{\tilde{g}}(w)$ for any $w \in W^{1,2}(B_1(0); N)$ with $v - w \in W_0^{1,2}(\overline{B_1(0)}; \mathbb{R}^n)$.

We assume the metric $\tilde{g}$ is C^2 on $\overline{B_1(0)}$. For $\Lambda > 0$ denote by $\mathscr{E}_\Lambda$ the class of functionals $E_{\tilde{g}}$ on $\overline{B_1(0)}$ with metric $\tilde{g}$ such that $\tilde{g}_{ij}(0) = \delta_{ij}$ and

$$\sum_{i,j,k} \left| \frac{\partial \tilde{g}_{ij}}{\partial x_k}(x) \right| \leq \Lambda.$$

If v is $E_{\tilde{g}}$-minimising with $E_{\tilde{g}} \in \mathscr{E}_\Lambda$ then we say $v \in \mathscr{H}_\Lambda$.

Schoen and Uhlenbeck [44] proved their ε-regularity theorem for minimisers of functionals of the form $\tilde{E}_{\tilde{g}} + F$, where F gives rise to terms in the Euler-Lagrange equations which are lower order than those coming from the energy. We state the result of their theorem with $F = 0$.

Lemma 3.6.0.1 (Theorem 3.1 in [44])**.** *There exists $\varepsilon = \varepsilon(m, N) > 0$ such that if $v \in \mathscr{H}_\Lambda$, $\Lambda \leq \varepsilon$ and $\int_{B_1(0)} |\nabla v|^2 \, \mathrm{d}x \leq \varepsilon$, then v is Hölder continuous in $B_{\frac{1}{2}}(0)$ and*

$$|v(x_1) - v(x_2)| \leq C|x_1 - x_2|^\gamma \tag{3.6.3}$$

for constants $C = C(m, N)$ and $\gamma = \gamma(m, N) \in (0, 1)$ and every $x_1, x_2 \in B_{\frac{1}{2}}(0)$.

Hölder continuous weakly harmonic maps are smooth. This is the content of the following lemma, proved by Schoen in [45].

Lemma 3.6.0.2 (Lemma 3.1 of [45])**.** *Consider a ball $B_r(y) \subset \mathbb{R}^{m+1}_+$ and suppose $v \in W^{1,2}(B_r(y); N)$ is a weakly harmonic map which is Hölder continuous on $B_r(y)$. Then v is smooth on $B_r(y)$.*

The final lemma we will need is from [45].

Lemma 3.6.0.3 (Theorem 2.2 of [45])**.** *Let $v \in C^2(B_r(0); N)$ and $\tilde{g}$ be a Riemannian metric on $B_r(0)$. Suppose v is harmonic with respect to $\tilde{g}$ in $B_r(0)$ and $\tilde{g}$ satisfies*

$$\left| \frac{\partial \tilde{g}_{ij}}{\partial x_k} \right| \leq \Lambda r^{-1} \tag{3.6.4}$$

for $i, j, k = 1, \ldots, m+1$ and $\Lambda^{-1}(\delta_{ij}) \leq (\tilde{g}_{ij}) \leq \Lambda(\delta_{ij})$ in the sense of tensors, where $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ otherwise. Then there exists an $\varepsilon = \varepsilon(\Lambda, m, N)$ such that if

$$r^{1-m} \int_{B_r(0)} |\nabla v|^2_{\tilde{g}} \left(det(\tilde{g}) \right)^{\frac{1}{2}} \mathrm{d}x \leq \varepsilon$$

then

$$\sup_{B_{\frac{r}{2}}(0)} |\nabla v|^2_{\tilde{g}} \leq C r^{-(1+m)} \int_{B_r(0)} |\nabla v|^2_{\tilde{g}} \left(det(\tilde{g}) \right)^{\frac{1}{2}} \mathrm{d}x \tag{3.6.5}$$

for a constant $C = C(\Lambda, m, N)$.

We are now in a position to prove an estimate of the form (3.6.1).

Lemma 3.6.0.4. *Let $\beta \in (-1, 1)$ and $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ be a minimiser of E^β relative to $\mathcal{O}$. Suppose $B^+_R(x_0)$ is a half-ball with $R \leq 1$ and $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$. There exists an $\varepsilon_0 = \varepsilon_0(m, N) > 0$, a $\theta = \theta(m, N) \geq 2$ and a positive $C = C(m, N)$ such that if*

$$R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \, \mathrm{d}x \leq \varepsilon_0,$$

then

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \mathrm{d}x \leq C \left(\frac{\rho}{r} \right)^\gamma r^{1-m} \int_{B_r(y)} |\nabla v|^2 \mathrm{d}x \tag{3.6.6}$$

on every $B_r(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$ for $0 < \rho \leq r$ and a $\gamma = \gamma(m, N) \in (0, 1)$.

Proof. Let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ be a minimiser of E^β relative to $\mathcal{O}$ and fix $B^+_R(x_0)$ with $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$ and $R \leq 1$. Our strategy for the proof is as follows. We show that provided the scaled energy $R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x$ is sufficiently small, v is Hölder continuous on every $B_r(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$ for a suitably chosen θ by using Lemma 3.6.0.1. It then follows from Lemma 3.6.0.2 that on every such ball v is smooth. Finally we show that we may apply Lemma 3.6.0.3 on every $B_r(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$ if the energy $R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x$ is sufficiently small. This allows us to deduce decay estimates of the form (3.5.15) for the scaled energy $e^{\rho C|\beta|} \rho^{1-m} \int_{B_\rho(y)} x^\beta_{m+1} |\nabla v|^2 \, \mathrm{d}x$, where $C = (y_{m+1} - \rho_0)^{-1}$ for an appropriately chosen ρ_0 as in the interior monotonicity formula, Lemma 3.3.2.1, and $\rho \leq r$. Then we may apply Lemma 3.5.1.1 to conclude the statement of the lemma.

Suppose $R^{1-m-\beta} \int_{B^+_R(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \leq \varepsilon_0$ for an $\varepsilon_0 > 0$ to be chosen small. First we will show, given this assumption, that the rescaled maps $v_{r,y}(x) = v(rx+y)$, defined for $x \in B_1(0)$ and r and y corresponding to $B_r(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$ where θ is to be chosen, satisfy the assumptions of Lemma 3.6.0.1. In particular, by defining appropriate metrics on $\overline{B_1(0)}$, we show that $v_{r,y} \in \mathscr{H}_\varepsilon$ and $\int_{B_1(0)} |\nabla v_{r,y}|^2 \leq \varepsilon$ where ε is the number from Lemma 3.6.0.1.

In order to show that $v_{r,y} \in \mathscr{H}_\varepsilon$ we must show that $v_{r,y}$ is a minimiser, in the sense of 3.6.0.1, of an energy on $B_1(0)$. To see this we introduce the following metrics on $\overline{B_1(0)}$. Recall that the metric g, defined by (3.0.1), is given in coordinates by $g_{ij}(x) = x^\alpha_{m+1} \delta_{ij}$ for $i, j = 1, \ldots, m+1$. Define $\hat{g}$ on $\overline{B_1(0)}$, given in Euclidean coordinates by

$$\hat{g}_{ij}(x) = \delta_{ij} \left(1 + \frac{r}{y_{m+1}} x_{m+1}\right)^\alpha . \tag{3.6.7}$$

The energy corresponding to $\hat{g}$ is

$$E_{\hat{g}}(\hat{v}) = \frac{1}{2} \int_{B_1(0)} \left(1 + \frac{r}{y_{m+1}} x_{m+1}\right)^\beta |\nabla \hat{v}|^2 \, \mathrm{d}x$$

for maps $\hat{v} \in W^{1,2}(B_1(0); \mathbb{R}^n)$.

First we consider bounds for the metrics $\hat{g}$ and use these to choose a first upper bound for θ. Notice that $\hat{g}$ and g are related as follows; for $x \in B_1(0)$ we have

$$\hat{g}_{ij}(x) = y^{-\alpha}_{m+1} g_{ij}(\rho x + y).$$

We could consider bounds for these metrics, directly applying the fact that $\frac{1}{2} \leq 1 + \frac{r}{y_{m+1}} x_{m+1} \leq \frac{3}{2}$ since $B_r(y) \in B_\theta(x_0, R, \frac{R}{3})$ for a $\theta \geq 2$ implies $B_r(y) \in \mathcal{B}$, that is, $2r \leq y_{m+1}$. However, we already have bounds for z^β_{m+1} when z is in such a $B_r(y)$, recall (3.4.3) in Section 3.4. Since $\beta = \alpha\left(\frac{m-1}{2}\right) \in (-1,1)$ this yields bounds for $\hat{g}$. In particular, assuming $x \in B_1(0)$ and $z \in B_r(y)$ are related by $z = rx + y$, it follows from (3.4.3) that there exist constants c, C depending only on m such that

$$c \leq \hat{g}_{ij}(x)^{\frac{m-1}{2}} = \delta_{ij}\left(1 + \frac{r}{y_{m+1}} x_{m+1}\right)^{\beta} = \delta_{ij} y_{m+1}^{-\beta} z_{m+1}^{\beta} = y_{m+1}^{-\beta} g_{ij}(z)^{\frac{m-1}{2}} \leq C \tag{3.6.8}$$

and

$$c \leq \hat{g}_{ij}(x) = y_{m+1}^{-\alpha} g_{ij}(z) \leq C. \tag{3.6.9}$$

In addition to the preceding bounds for $\hat{g}$, we consider bounds for its derivatives; this gives a lower bound for θ. We find

$$\frac{\partial}{\partial x_i}\left(1 + \frac{r}{y_{m+1}} x_{m+1}\right)^{\alpha} = 0$$

for $i = 1, \ldots, m$. Moreover, using (3.6.9) and the fact that $\frac{1}{2} \leq 1 + \frac{r}{y_{m+1}} x_{m+1} \leq \frac{3}{2}$, we have

$$\begin{aligned}\left|\frac{\partial}{\partial x_{m+1}}\left(1 + \frac{r}{y_{m+1}} x_{m+1}\right)^{\alpha}\right| &= \frac{r}{y_{m+1}} |\alpha| \left|\left(1 + \frac{r}{y_{m+1}} x_{m+1}\right)^{\alpha - 1}\right| \\ &\leq C \frac{r}{y_{m+1}} \end{aligned} \tag{3.6.10}$$

where C is chosen independently of α. Now suppose

$$r \leq \min\left\{\frac{y_{m+1}\varepsilon}{(m+1)C}, \frac{y_{m+1}}{2}\right\} \tag{3.6.11}$$

where C is from (3.6.10) and ε is the constant given by Lemma 3.6.0.1. Then we have

$$\sum_{i,j,k}\left|\frac{\partial \tilde{g}_{ij}}{\partial x_k}\right| = \sum_{i=1}^{m+1}\left|\frac{\partial \tilde{g}_{ii}}{\partial x_{m+1}}\right| \leq \varepsilon. \tag{3.6.12}$$

Thus we choose

$$\theta = \theta(m, N) \geq \max\left\{2, \frac{(m+1)C}{\varepsilon}\right\} \tag{3.6.13}$$

and assume $B_r(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$ henceforth, this implies (3.6.11) and hence (3.6.12) holds on any such $B_r(y)$.

In order to conclude that $v_{r,y} \in \mathscr{H}_\varepsilon$ for every $B_r(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$, it remains to show that if $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ is a minimiser of E^β in the sense of definition 3.2.0.1, then the map $v_{r,y}(x) = v(rx + y)$ satisfies $v_{r,y} \in W^{1,2}(B_1(0); N)$ and is an energy minimiser of $E_{\hat{g}}$ in the sense of definition 3.6.0.1. A change of variables gives

$$\int_{B_1(0)} \left(1 + \frac{r}{y_{m+1}} x_{m+1}\right)^{\beta} |\nabla v_{r,y}|^2 \, \mathrm{d}x = y_{m+1}^{-\beta} r^{1-m} \int_{B_r(y)} x_{m+1}^{\beta} |\nabla v|^2 \, \mathrm{d}x \quad (3.6.14)$$

and

$$\int_{B_1(0)} \left(1 + \frac{r}{y_{m+1}} x_{m+1}\right)^{\beta} |v_{r,y}|^2 \, \mathrm{d}x = y_{m+1}^{-\beta} r^{-(1+m)} \int_{B_r(y)} x_{m+1}^{\beta} |v|^2 \, \mathrm{d}x. \quad (3.6.15)$$

Thus, noting that as a consequence of Lemma 2.2.1.1 we have $\dot{W}^{1,2}(\mathbb{R}_+^{m+1}; N) \hookrightarrow W^{1,2}(B_r(y); N)$ for every $B_r(y)$ with $\overline{B_r(y)} \subset \mathbb{R}_+^{m+1}$, regardless of $\beta \in (-1, 1)$, using (3.6.8) we conclude from (3.6.14) and (3.6.15) that $v_{r,y} \in W^{1,2}(B_1(0); N)$.

Now we show that $v_{r,y}$ is a minimiser of $E_{\hat{g}}$ in the sense of 3.6.0.1. Let $w \in W^{1,2}(B_1(0); N)$ be such that $v_{r,y} - w \in W_0^{1,2}(B_1(0); \mathbb{R}^n)$. We will use this w to define a map in $\dot{W}_{\beta}^{1,2}(\mathbb{R}_+^{m+1}; N)$ to which we may apply definition 3.2.0.1. Notice that the map $\hat{w}(z) = w\left(\frac{z-y}{r}\right)$, for $z \in B_r(y)$, satisfies $\hat{w} \in W^{1,2}(B_r(y); N)$; w can be approximated in $W^{1,2}(B_1(0); \mathbb{R}^n)$ by a sequence in $C^{\infty}(B_1(0); \mathbb{R}^n) \cap W^{1,2}(B_1(0); \mathbb{R}^n)$ and composing the elements of this sequence with the map $z \to \frac{z-y}{r}$, we obtain a sequence in $C^{\infty}(B_r(y); \mathbb{R}^n) \cap W^{1,2}(B_r(y); \mathbb{R}^n)$ which approximates $\hat{w}$ in $W^{1,2}(B_r(y); \mathbb{R}^n)$. We extend the function $\hat{w}(z) = w\left(\frac{z-y}{r}\right)$, for $z \in B_r(y)$, to $\mathbb{R}_+^{m+1}$ by requiring $\hat{w} = v$ outside $B_r(y)$. It follows from the fact that $\hat{w}(z) = w\left(\frac{z-y}{r}\right) \in N$ for almost every $z \in B_r(y)$ and $\hat{w}(z) = v(z) \in N$ for almost every $z \in \mathbb{R}_+^{m+1} \backslash B_r(y)$, that $\hat{w}(z) \in N$ for almost every $z \in \mathbb{R}_+^{m+1}$. Thus, to conclude $\hat{w} \in \dot{W}_{\beta}^{1,2}(\mathbb{R}_+^{m+1}; N)$, we need to construct a sequence in $\mathcal{D}_+(\mathbb{R}_+^{m+1}; \mathbb{R}^n)$ which converges to $\hat{w}$ in $\dot{W}_{\beta}^{1,2}(\mathbb{R}_+^{m+1}; \mathbb{R}^n)$. Let $(\phi_k)_{k \in \mathbb{N}}$ be a sequence in $C_0^{\infty}(B_r(y); \mathbb{R}^n)$ such that $\phi_k \to \hat{w} - v$ in $W_0^{1,2}(B_r(y); \mathbb{R}^n)$. Without relabelling, we extend the domain of each ϕ_k to $\mathbb{R}_+^{m+1}$ by requiring $\phi_k = 0$ in $\mathbb{R}_+^{m+1} \backslash B_r(y)$. Then every $\phi_k \in C_0^{\infty}(\mathbb{R}_+^{m+1}; \mathbb{R}^n)$. Furthermore, let $(\psi_k)_{k \in \mathbb{N}}$, with $\psi_k \in \mathcal{D}_+(\mathbb{R}_+^{m+1}; \mathbb{R}^n)$ for each k, be a sequence such that $\psi_k \to v$ in $\dot{W}_{\beta}^{1,2}(\mathbb{R}_+^{m+1}; \mathbb{R}^n)$. It follows that $\phi_k + \psi_k \in \mathcal{D}_+(\mathbb{R}_+^{m+1}; \mathbb{R}^n)$ for every k. We calculate

$$\begin{aligned}\int_{\mathbb{R}_+^{m+1}} x_{m+1}^{\beta} |\nabla(\phi_k + \psi_k - (\hat{w} - v + v)|^2 \mathrm{d}x \leq 2 \int_{B_r(y)} x_{m+1}^{\beta} |\nabla \phi_k - \nabla(\hat{w} - v)|^2 \mathrm{d}x \\ + 2 \int_{\mathbb{R}_+^{m+1}} x_{m+1}^{\beta} |\nabla \psi_k - v|^2 \mathrm{d}x\end{aligned}$$

and observe that the right hand side of the above tends to 0 as $k \to \infty$. Thus

$\phi_k + \psi_k \to \hat{w}$ in $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ and $\hat{w} = v$ outside $B_r(y)$ so we may compare the E^β energies of $\hat{w}$ and v.

A change of variables gives

$$\begin{aligned}\int_{B_1(0)} &\left(1 + \frac{r}{y_{m+1}} x_{m+1}\right)^\beta |\nabla w|^2 \,\mathrm{d}x \\ &= y_{m+1}^{-\beta} r^{1-m} \int_{B_r(y)} z_{m+1}^\beta \left|\nabla \left(w\left(\frac{z-y}{r}\right)\right)\right|^2 \mathrm{d}z \end{aligned} \tag{3.6.16}$$

and

$$\int_{B_1(0)} \left(1 + \frac{r}{y_{m+1}} x_{m+1}\right)^\beta |\nabla (v(rx+y))|^2 \,\mathrm{d}x = y_{m+1}^{-\beta} r^{1-m} \int_{B_r(y)} z_{m+1}^\beta |\nabla v|^2 \,\mathrm{d}z. \tag{3.6.17}$$

Since v is an energy minimiser for E^β relative to $\mathcal{O}$, $\hat{w} = v$ outside $B_r(y)$ and $v, \hat{w} \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ we have

$$\int_{\mathbb{R}^{m+1}_+} z_{m+1}^\beta |\nabla v|^2 \,\mathrm{d}z \le \int_{\mathbb{R}^{m+1}_+} z_{m+1}^\beta |\nabla \hat{w}|^2 \,\mathrm{d}z. \tag{3.6.18}$$

Then, as $\hat{w} = v$ outside $B_r(y)$, subtracting $\int_{\mathbb{R}^{m+1}_+ \setminus B_r(y)} z_{m+1}^\beta |\nabla v|^2 \,\mathrm{d}z$ from (3.6.18) gives

$$\int_{B_r(y)} z_{m+1}^\beta |\nabla v|^2 \,\mathrm{d}z \le \int_{B_r(y)} z_{m+1}^\beta |\nabla \hat{w}|^2 \,\mathrm{d}z$$

and thus

$$y_{m+1}^{-\beta} r^{1-m} \int_{B_r(y)} z_{m+1}^\beta |\nabla v|^2 \,\mathrm{d}z \le y_{m+1}^{-\beta} r^{1-m} \int_{B_r(y)} z_{m+1}^\beta \left|\nabla \left(w\left(\frac{z-y}{r}\right)\right)\right|^2 \mathrm{d}z. \tag{3.6.19}$$

Hence, transforming both sides of (3.6.19) back to integrals over $B_1(0)$ using (3.6.16) and (3.6.17), we find

$$\int_{B_1(0)} \left(1 + \frac{r}{y_{m+1}} x_{m+1}\right)^\beta |\nabla v_{r,y}|^2 \,\mathrm{d}x \le \int_{B_1(0)} \left(1 + \frac{r}{y_{m+1}} x_{m+1}\right)^\beta |\nabla w|^2 \,\mathrm{d}x$$

which shows that $v_{r,y}$ is a minimiser in the sense of definition 3.6.0.1 on $B_1(0)$.

We have shown so far that if $B_r(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$, for θ as in (3.6.13), then $v_{r,y} \in \mathscr{H}_\varepsilon$. Thus, as soon as $\int_{B_1(0)} |\nabla v_{r,y}|^2 \,\mathrm{d}x \le \varepsilon$, Lemma 3.6.0.1 applies. We can, however, guarantee this is true by applying Lemma 3.4.0.2. We have, changing variables,

$$\int_{B_1(0)} |\nabla v_{r,y}|^2 \,\mathrm{d}x = r^{1-m} \int_{B_r(y)} |\nabla v|^2 \,\mathrm{d}x.$$

Then Lemma 3.4.0.2 yields

$$r^{1-m} \int_{B_r(y)} |\nabla v|^2 \,\mathrm{d}x \leq CR^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^{\beta} |\nabla v|^2 \,\mathrm{d}x \tag{3.6.20}$$

for a positive constant $C = C(m)$. Hence, if we assume $\varepsilon_0 \leq \frac{\varepsilon}{C}$ where C is the constant from (3.6.20), then

$$\int_{B_1(0)} |\nabla v_{r,y}|^2 \,\mathrm{d}x \leq CR^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^{\beta} |\nabla v|^2 \,\mathrm{d}x \leq C\varepsilon_0 = \varepsilon \tag{3.6.21}$$

for every r, y corresponding to a ball $B_r(y) \in \mathcal{B}_\theta\left(x_0, R, \frac{R}{3}\right)$. Thus we choose $\varepsilon_0 = \varepsilon_0(m, N) = \frac{\varepsilon}{C}$. Consequently, $v_{r,y} \in \mathscr{H}_\varepsilon$ and satisfies the conditions of lemma 3.6.0.1 for every such ball. An application of this lemma guarantees the existence of a $C = C(m, N)$ and a $\gamma = \gamma(m, N) \in (0, 1)$ such that

$$|v_{r,y}(x_1) - v_{r,y}(x_2)| \leq C|x_1 - x_2|^\gamma \tag{3.6.22}$$

for every $x_1, x_2 \in B_{\frac{1}{2}}(0)$. Let $z_1, z_2 \in B_r(y)$. Then $z_i = rx_i + y$ for some $x_1, x_2 \in B_{\frac{1}{2}}(0)$ and for $i = 1, 2$. It follows that

$$|v(z_1) - v(z_2)| = |v_{r,y}(x_1) - v_{r,y}(x_2)| \leq C|x_1 - x_2|^\gamma = \frac{C}{r^\gamma}|z_1 - z_2|^\gamma. \tag{3.6.23}$$

Hence v is Hölder continuous in every $B_{\frac{r}{2}}(y) \in \mathcal{B}_\theta(x_0, R, \frac{R}{3})$. Equivalently, v is Hölder continuous in every $B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$.

Since v is weakly harmonic in $\mathbb{R}^{m+1}_+$, as defined in Chapter 1 and discussed in Section 3.1.1, it is weakly harmonic on every $B_r(y)$ with $\overline{B_r(y)} \subset \mathbb{R}^{m+1}_+$. Thus it follows from Lemma 3.6.0.2 that v is smooth in each $B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$, which holds if and only if each $v_{r,y}$ corresponding to such a $B_r(y)$ is smooth in $B_1(0)$.

Our aim is now to apply Lemma 3.6.0.3 to yield decay estimates of the form (3.5.15) for the energy of $v_{r,y}$ on $B_1(0)$. Let $\tilde{\Lambda} = \tilde{\Lambda}(m) = \max\{\varepsilon, c, c^{-1}, C, C^{-1}\}$ where c and C are the constants from (3.6.9) and ε is as above. It then follows from (3.6.9) and (3.6.12) that the metrics $\hat{g}$ satisfy (3.6.4) and $\tilde{\Lambda}^{-1}(\delta_{ij}) \leq (\hat{g}_{ij}) \leq \tilde{\Lambda}(\delta_{ij})$ as in Lemma 3.6.0.3 on $B_1(0)$ with this choice of $\tilde{\Lambda}$. Furthermore, as v is smooth in $B_r(y)$, it is a smooth harmonic map with respect to g satisfying (3.6.2) on this set. An integration by parts in (3.6.2) shows that

$$\operatorname{div}(z_{m+1}^{\beta}\nabla v) + z_{m+1}^{\beta}A(v)(\nabla v, \nabla v) = 0 \text{ in } B_r(y).$$

We multiply this expression by $y_{m+1}^{-\beta}$ and apply the chain rule to see that $v_{r,y}$ satisfies

$$\operatorname{div}\left(\left(1+\frac{r}{y_{m+1}}x_{m+1}\right)^{\beta}\nabla v_{r,y}\right) + \left(1+\frac{r}{y_{m+1}}x_{m+1}\right)^{\beta}A(v_{r,y})(\nabla v_{r,y}, \nabla v_{r,y}) = 0$$

in $B_1(0)$. This implies $v_{r,y}$ is a smooth harmonic map with respect to $\hat{g}$ in $B_1(0)$ which satisfies the assumptions of Lemma 3.6.0.3. Thus the lemma gives an $\varepsilon_1 = \varepsilon_1(\Lambda, m, N) = \varepsilon_1(m, N)$ such that if

$$\int_{B_1(0)}\left(1+\frac{r}{y_{m+1}}x_{m+1}\right)^{\beta}|\nabla v_{r,y}|^2\mathrm{d}x \leq \varepsilon_1$$

then (3.6.5) applies. We combine (3.6.8) with (3.6.21) to see that

$$\int_{B_1(0)}\left(1+\frac{r}{y_{m+1}}x_{m+1}\right)^{\beta}|\nabla v_{r,y}|^2\mathrm{d}x \leq C\int_{B_1(0)}|\nabla v_{r,y}|^2\mathrm{d}x \leq C\varepsilon_0. \qquad (3.6.24)$$

Thus, assuming $\varepsilon_0 \leq \frac{\varepsilon_1}{C}$, it follows that we may apply Lemma 3.6.0.3. We do so, recalling that $\frac{1}{2} \leq 1 + \frac{r}{y_{m+1}}x_{m+1} \leq \frac{3}{2}$ and $\beta = \alpha\frac{m+1}{2} - \alpha$, and see that

$$\begin{aligned}
r^2\sup_{B_{\frac{r}{2}}(y)}\left(\frac{x_{m+1}}{y_{m+1}}\right)^{\beta}|\nabla v|^2 &= \sup_{B_{\frac{1}{2}}(0)}\left(1+\frac{r}{y_{m+1}}x_{m+1}\right)^{\beta}|\nabla v_{r,y}|^2 \\
&\leq C\sup_{B_{\frac{1}{2}}(0)}\left(1+\frac{r}{y_{m+1}}x_{m+1}\right)^{-\alpha}|\nabla v_{r,y}|^2 \\
&\leq C\int_{B_1(0)}\left(1+\frac{r}{y_{m+1}}x_{m+1}\right)^{\beta}|\nabla v_{r,y}|^2\mathrm{d}x \\
&= r^{1-m}\int_{B_r(y)}\left(\frac{x_{m+1}}{y_{m+1}}\right)^{\beta}|\nabla v|^2\mathrm{d}x.
\end{aligned}$$

As a result, for any $\sigma \in (0, \frac{1}{2}]$ we have

$$(\sigma r)^{1-m} \int_{B_{\sigma r}(y)} \left(\frac{x_{m+1}}{y_{m+1}}\right)^\beta |\nabla v|^2 \mathrm{d}x \leq C(\sigma r)^2 r^{-(1+m)} \int_{B_r(y)} \left(\frac{x_{m+1}}{y_{m+1}}\right)^\beta |\nabla v|^2 \mathrm{d}x$$
$$= C\sigma^2 r^{1-m} \int_{B_r(y)} \left(\frac{x_{m+1}}{y_{m+1}}\right)^\beta |\nabla v|^2 \mathrm{d}x.$$

Choose σ such that $\sigma^2 \leq \frac{1}{2C}$ and notice that for $\rho \leq r$, the scaling factor $e^{\frac{|\beta|\rho}{y_{m+1}-r}}$ in Lemma 3.3.2.1 is increasing in ρ. It follows that

$$\begin{aligned} &e^{\frac{|\beta|\sigma r}{y_{m+1}-r}} (\sigma r)^{1-m} \int_{B_{\sigma r}(y)} \left(\frac{x_{m+1}}{y_{m+1}}\right)^\beta |\nabla v|^2 \mathrm{d}x \\ &\leq \frac{1}{2} e^{\frac{|\beta| r}{y_{m+1}-r}} r^{1-m} \int_{B_{\sigma r}(y)} \left(\frac{x_{m+1}}{y_{m+1}}\right)^\beta |\nabla v|^2 \mathrm{d}x \end{aligned}$$

on every $B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$. Furthermore, Lemma 3.3.2.1 implies that the map $\rho \mapsto e^{\frac{|\beta|\rho}{y_{m+1}-r}} \rho^{1-m} \int_{B_\rho(y)} \left(\frac{x_{m+1}}{y_{m+1}}\right)^\beta |\nabla v|^2 \mathrm{d}x$ is non-decreasing in ρ for $\rho \leq r$. We may thus apply Lemma 3.5.1.1. We find

$$\begin{aligned} &e^{\frac{|\beta|\rho}{y_{m+1}-r}} \rho^{1-m} \int_{B_\rho(y)} \left(\frac{x_{m+1}}{y_{m+1}}\right)^\beta |\nabla v|^2 \mathrm{d}x \\ &\leq C \left(\frac{\rho}{r}\right)^\gamma e^{\frac{|\beta| r}{y_{m+1}-r}} r^{1-m} \int_{B_r(y)} \left(\frac{x_{m+1}}{y_{m+1}}\right)^\beta |\nabla v|^2 \mathrm{d}x \end{aligned} \tag{3.6.25}$$

for every $\rho \leq r$ on every $B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$. Notice that

$$e^{\frac{|\beta|(r-\rho)}{y_{m+1}-r}} = e^{\frac{|\beta|(r-y_{m+1}+y_{m+1}-\rho)}{y_{m+1}-r}} \leq e$$

since $y_{m+1} - r \geq \frac{y_{m+1}}{2}$ and $y_{m+1} - \rho \leq y_{m+1}$ which follows as $B_\rho(y) \subset B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$. Using this fact, combined with (3.4.3) and (3.6.25), we deduce that

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \mathrm{d}x \leq C \left(\frac{\rho}{r}\right)^\gamma r^{1-m} \int_{B_r(y)} |\nabla v|^2 \mathrm{d}x$$

on every $B_r(y) \in \mathcal{B}_{2\theta}(x_0, R, \frac{R}{3})$ for $0 < \rho \leq r$. This concludes the proof. □

This lemma gives us a condition on the energy E^β such that minimisers essentially satisfy (3.5.4) in Lemma 3.5.0.2 and comprises the first milestone in the proof of our ε-regularity theorem.

3.7 An Overview of the Boundary Estimates for Hölder Continuity

Our strategy to prove that energy minimisers satisfy (3.5.3) in Lemma 3.5.0.2, which will complete our proof of Hölder continuity of minimisers of E^β relative to $\mathcal{O}$, consists of proving three main estimates. First we construct a comparison function w which satisfies

$$\begin{aligned} &\sigma^{1-m-\beta}\int_{B^+_\sigma(y)} x^\beta_{m+1}|\nabla w|^2\mathrm{d}x \\ &\leq \varepsilon\rho^{1-m-\beta}\int_{B^+_\rho(y)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x + \frac{1}{\varepsilon}\rho^{-(1+m+\beta)}\int_{B^+_\rho(y)} x^\beta_{m+1}|v-\overline{v}_{B^+_\rho(y),\beta}|^2\mathrm{d}x \end{aligned} \tag{3.7.1}$$

where $\varepsilon \in (0,1)$, $\sigma \in \left(\frac{3\rho}{4},\rho\right)$ and $y \in \partial\mathbb{R}^{m+1}_+$. This function is constructed in such a way that we may compare its energy with the energy of a minimiser of E^β relative to $\mathcal{O}$. If v is such a minimising harmonic map then we will show that as a result of the monotonicity formula, Lemma 3.3.1.1, and the assumption of small energy, we have good control over the first term on the right hand side of (3.7.1). In order to control the other term we prove an improved version of the Poincaré inequality of the form

$$(\theta\rho)^{-(1+m+\beta)}\int_{B^+_{\theta\rho}(y)} x^\beta_{m+1}|v-\overline{v}_{B^+_\rho(y),\beta}|^2\mathrm{d}x \leq \delta\rho^{1-m-\beta}\int_{B^+_\rho(y)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x, \tag{3.7.2}$$

where $\theta \in (0,1)$ depends on δ in addition to some other factors.

Combining (3.7.1) and (3.7.2) we prove an energy decay estimate on concentric half-balls. In particular we show

$$(\theta_0\rho)^{1-m-\beta}\int_{B^+_{\theta_0\rho}(y)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x \leq \frac{1}{2}\rho^{1-m-\beta}\int_{B^+_\rho(y)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x \tag{3.7.3}$$

for a $\theta_0 \in (0,\frac{1}{4})$. This allows us to show that a minimising harmonic map satisfies (3.5.3) in Lemma 3.5.0.2 as we require. The combination of this fact with Lemma 3.6.0.4 will allow us to conclude that energy minimisers are Hölder continuous.

3.8 A Modified Lemma of Luckhaus

In order to show that (3.7.1) holds we first prove a modified version of a lemma of Luckhaus, Lemma 3 in [29], as presented in Lemma 1 Section 2.6 of [46]. This is the main step in constructing the comparison function w as described in Section 3.7. A consequence of the lemma we prove, and an analogue of Corollary 1 in Section 2.7 of [46], is an estimate of the form (3.7.1).

Let $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ denote the m dimensional unit sphere, centred at the origin and equipped with the metric induced by the Euclidean metric on $\mathbb{R}^{m+1}$. Define $\mathbb{S}^m_+ = \mathbb{S}^m \cap \mathbb{R}^{m+1}_+$ with the metric induced from $\mathbb{S}^m$. We let ω denote a point in $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ or $\mathbb{S}^m_+ \subset \mathbb{R}^{m+1}_+$and write $d\omega$ for the volume element corresponding to the induced metric. We continue to use the notation $\partial^+\Omega = \partial\Omega \cap \mathbb{R}^{m+1}_+$ for $\Omega \subset \mathbb{R}^{m+1}$ and $Q_r(y) = \{x \in \mathbb{R}^{m+1} : |x_i - y_i| < r, i = 1, \dots, m+1\}$ for $y \in \mathbb{R}^{m+1}$. We also write $Q^+_r(y) = Q_r(y) \cap \mathbb{R}^{m+1}_+$ for $y \in \mathbb{R}^m \times \{0\}$.

In order to state the modified Luckhaus lemma precisely we introduce the notion of a Sobolev space for functions whose domain is either $\mathbb{S}^m$ or $\mathbb{S}^m_+$. There are several possible definitions of this space and we have chosen the one most congruous with our methodology. The role the following Sobolev space plays will become evident during the proof of the lemma, it is connected with the notion of homogeneous degree zero extensions of functions which we discuss below in Section 3.8.1.

Definition 3.8.0.1. Let $\varepsilon > 0$ and $\rho > 0$. Suppose $S = \rho\mathbb{S}^m$ and $V_\varepsilon = B_{\rho+\varepsilon}(0)\backslash B_{\rho-\varepsilon}(0)$ or $S = \rho\mathbb{S}^m_+$ and $V_\varepsilon = B^+_{\rho+\varepsilon}(0)\backslash B^+_{\rho-\varepsilon}(0)$. An element $v \in L^2_\beta(S;\mathbb{R}^n)$ is said to be in $W^{1,2}_\beta(S;\mathbb{R}^n)$ if the map $v(\rho\frac{x}{|x|}) \in W^{1,2}_\beta(V_\varepsilon;\mathbb{R}^n)$ for some $\varepsilon > 0$. An element $v \in L^2_\beta(S \times [a,b];\mathbb{R}^n)$, with $a < b$ real numbers, is said to be in $W^{1,2}_\beta(S \times [a,b];\mathbb{R}^n)$ if the map $v(\rho\frac{x}{|x|}, s) \in W^{1,2}_\beta(V_\varepsilon \times [a,b];\mathbb{R}^n)$ for some $\varepsilon > 0$. If $N \subset \mathbb{R}^n$ is compact, we say v is in $W^{1,2}_\beta(S;N)$ or $W^{1,2}_\beta(S \times [a,b];N)$ if v is in $W^{1,2}_\beta(S;\mathbb{R}^n)$ or $W^{1,2}_\beta(S \times [a,b];\mathbb{R}^n)$ respectively and $v(x) \in N$ for almost every $x \in S$.

Lemma 3.8.0.1. *Let $m + 1 \geq 3$ and $\beta \in (-1,1)$. Let N be a compact subset of $\mathbb{R}^n$ and suppose $u, v \in W^{1,2}_\beta(\mathbb{S}^m_+;N)$. Then for all $\varepsilon \in (0,1)$ there is a $w \in W^{1,2}_\beta(\mathbb{S}^m_+ \times [0,\varepsilon];\mathbb{R}^n)$ such that w agrees with u on $\mathbb{S}^m_+ \times \{0\}$ and v on $\mathbb{S}^m_+ \times \{\varepsilon\}$ in the sense of traces and which satisfies the following. Let $\overline{D}$ be the gradient on*

$\mathbb{S}^m_+ \times [0,\varepsilon]$ *and* D *the gradient on* $\mathbb{S}^m_+$. *Then* $w = w(\omega, s)$ *satisfies*

$$\begin{aligned}
&\int_{\mathbb{S}^m_+\times[0,\varepsilon]} \omega_{m+1}^{\beta}|\overline{D}w|^2 \mathrm{d}\omega\mathrm{d}s \\
&\leq C_1\varepsilon \int_{\mathbb{S}^m_+} \omega_{m+1}^{\beta}\left(|Du|^2+|Dv|^2\right)\mathrm{d}\omega + \frac{C_1}{\varepsilon}\int_{\mathbb{S}^m_+}\omega_{m+1}^{\beta}|u-v|^2\mathrm{d}\omega
\end{aligned} \tag{3.8.1}$$

where $C_1 = C_1(m,\beta)$. *Furthermore,* w *satisfies*

$$\begin{aligned}
&dist^2(w(\omega,s),N) \\
&\leq \frac{C_2}{\varepsilon^{m+\frac{\beta}{2}+\frac{|\beta|}{2}}}\left(\int_{\mathbb{S}^m_+}\omega_{m+1}^{\beta}\left(|Du|^2+|Dv|^2\right)\mathrm{d}\omega\right)^{\frac{1}{q}}\left(\int_{\mathbb{S}^m_+}\omega_{m+1}^{\beta}|u-v|^2\mathrm{d}\omega\right)^{1-\frac{1}{q}} \\
&\quad + \frac{C_2}{\varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}}\int_{\mathbb{S}^m_+}\omega_{m+1}^{\beta}|u-v|^2\mathrm{d}\omega
\end{aligned} \tag{3.8.2}$$

for almost every $(\omega, s) \in \mathbb{S}^m_+ \times [0,\varepsilon]$ *where* $C_2 = C_2(m,\beta)$ *and* q *satisfies the following. If* $\beta \in (-1,0]$ *then* (3.8.2) *holds for* $q = 2$ *and if* $\beta \in (0,1)$ *then* (3.8.2) *holds with either* $q = 2$ *or* $q = p$ *for any fixed* $p \in (1, \frac{2}{1+\beta})$.

Our proof of Lemma 3.8.0.1 follows the proof, given in section 2.12.2 of [46], of lemma 1 in section 2.6 of [46]. There are several preliminary results we will need for the proof and we proceed to discuss these.

3.8.1 Homogeneous Degree Zero Extension

We will use homogeneous degree zero extension to define functions on the interior of sets given their boundary values. Our primary use of this technique will be to inductively define functions on cubes of dimension $2,\ldots,m+2$; we start with a function defined on the l-dimensional boundary of an $l+1$ dimensional cube, for $l = 2,\ldots,m+1$, and use homogeneous degree zero extension to define a function on the whole cube. The benefit of this method of extension is that certain bounds obtained on boundary of the cubes are preserved when the map is extended into the interior. Here we discuss properties of maps which are homogeneous of degree zero with a view to applying the aforementioned method of homogeneous extension in the proof of Lemma 3.8.0.1.

The distributional derivative of a function v, which is homogeneous of degree zero with respect to a point $y \in \mathbb{R}^{l+1}$, can be expressed in terms of the derivative of the restriction of v to the boundaries of balls, half-balls, cubes or half-cubes centred at y, depending on the circumstances. Let $B_\rho = \overline{B^{l+1}_\rho(y)}$ and $Q_\rho =$

$\overline{Q_\rho^{l+1}(y)}$. Furthermore, for $y \in \mathbb{R}^l \times \{0\}$, we write $B_\rho^+ = B_\rho \cap \mathbb{R}_+^{l+1}$ and $Q_\rho^+ = Q_\rho \cap \mathbb{R}_+^{l+1}$ and recall the notation $\partial^+\Omega = \partial\Omega \cap \mathbb{R}_+^{l+1}$ for $\Omega \subset \mathbb{R}^{l+1}$.

We regard B_ρ, Q_ρ, ∂B_ρ and ∂Q_ρ as submanifolds of $\mathbb{R}^{l+1}$ with metrics induced by the Euclidean metric on $\mathbb{R}^{l+1}$. These are then submanifolds of at least Lipschitz regularity and thus admit gradient operators.

To further facilitate our discussion, we introduce the following notation in $\mathbb{R}^{l+1}$. Let $r = |x|$ be the radial variable and suppose $\omega \in \mathbb{S}^l$, the unit sphere in $\mathbb{R}^{l+1}$ centred at 0. Furthermore, let $\mathrm{d}\omega$ denote the volume form on $\mathbb{S}^l$ with respect to the metric induced from $\mathbb{R}^{l+1}$.

We will perform the following calculations on $B_1 = B$ and $Q_1 = Q$, centred at $y = 0$, the inequalities on B_ρ and Q_ρ then follow by rescaling. Suppose v is homogeneous of degree zero with respect to 0. We now discuss the relationship between the gradient of v on B with the gradient of $v|_{\mathbb{S}^l}$ on $\mathbb{S}^l$. We denote a gradient taken on $\mathbb{S}^l$ by $\hat{\nabla}$ and for simplicity we will also denote $v|_{\mathbb{S}^l}$ by v. Observe that $v(x) = v(r\omega) = v(\omega)$ for $\omega(x) = \frac{x}{r}$. It follows that $\frac{\partial v}{\partial r} \equiv 0$ and hence we have the identification

$$\nabla v = \hat{\nabla} v \tag{3.8.3}$$

on $\mathbb{S}^l$. Furthermore, we calculate

$$\begin{aligned} \nabla v_i(x) &= \nabla\left(v_i\left(\omega(x)\right)\right) \\ &= \frac{1}{|x|}\nabla v_i(\omega(x)) - \frac{x}{|x|^2}\frac{\partial v_i}{\partial r}(\omega(x)) \\ &= \frac{1}{|x|}\nabla v_i(\omega(x)) \\ &= \frac{1}{|x|}\hat{\nabla} v_i(\omega(x)) \end{aligned} \tag{3.8.4}$$

for $i = 1, \ldots, n$ and $x \in \mathbb{R}^{l+1}\backslash\{0\}$.

We observe a similar relationship between the gradient of v on Q and the gradient of its restriction to ∂Q. Let $\zeta \in \partial Q$ and define $\zeta(x) = \frac{x}{\max_{j=1,\ldots,l+1}|x_j|}$. Let $x \in \mathbb{R}^{l+1}\backslash\{0\}$ and suppose $\max_{j=1,\ldots,l+1}|x_j| = |x_k|$. Since v is homogeneous of degree zero with respect to 0 we have $v(x) = v(\zeta(x))$. Thus, for $i \neq k$ we calculate

$$\begin{aligned} \frac{\partial v}{\partial x_i}(x) &= \frac{\partial}{\partial x_i}(v(\zeta(x))) \\ &= \frac{\partial v}{\partial x_i}(\zeta(x))\frac{1}{|x_k|}. \end{aligned} \tag{3.8.5}$$

When $i = k$ we find

$$\frac{\partial v}{\partial x_k}(x) = -\mathrm{sgn}(x_k) \sum_{\substack{i=1 \\ i \neq k}}^{l+1} \frac{\partial v}{\partial x_i}(\zeta(x)) \frac{x_i}{|x_k|^2}. \tag{3.8.6}$$

Let $\tilde{\nabla}$ denote the gradient on the faces F of ∂Q. It follows from (3.8.5) and (3.8.6) that

$$|\nabla v| \leq C |\tilde{\nabla} v|, \tag{3.8.7}$$

on every face F of ∂Q, where $C = C(l)$.

Now we show that if v is homogeneous of degree zero with respect to 0 then integrals of the gradient of v over Q_ρ or Q_ρ^+ can be expressed in terms of integrals of the gradient on the respective boundaries. We describe the details of this relationship on half-cubes Q_ρ^+ and then state the corresponding result for cubes Q_ρ.

Henceforth we denote any l dimensional face of Q_ρ^+ which has no edges in the $l+1$ direction by F^l. We denote the collection of all such faces, excluding those which do not intersect $\mathbb{R}_+^{l+1}$, by $\mathcal{F}^l$. Furthermore, we write F_{l+1}^l for an l dimensional face of Q_ρ^+ that has edges in the $l+1$ direction and call the collection of these faces $\mathcal{F}_{l+1}^l$. Then

$$\partial^+ Q_\rho^+ = \bigcup_{\mathcal{F}^l} F^l \cup \bigcup_{\mathcal{F}_{l+1}^l} F_{l+1}^l.$$

In the proof of Lemma 3.8.0.1 we will regard the half-cubes Q_ρ^+ as subsets (submanifolds) of $\mathbb{R}_+^{m+1} \times \mathbb{R}$. To facilitate the compatibility of our considerations in the proof of the lemma with the discussion here, we consider the following integrals with respect to the $l+1$ dimensional Hausdorff measure $\mathrm{d}\mathcal{H}^{l+1}$ on $\mathbb{R}^{l+1}$.

First we consider $Q^+ = Q_1^+$. The general case will follow by rescaling. We transform the integral of $x_{l+1}^\beta |\nabla v|^2$ over Q^+ into an integral over $B^+ = B_1^+$. Recall the bi-Lipschitz, piecewise C^1 map $\Phi_y : \mathbb{R}^{l+1} \to \mathbb{R}^{l+1}$ with bi-Lipschitz, piecewise C^1 inverse $\Phi_y^{-1} : \mathbb{R}^{l+1} \to \mathbb{R}^{l+1}$ defined in section 2.3.1. When $y = 0$, this map is given by $\Phi_0(x) = |x_k| \frac{x}{|x|}$ if $x \neq 0$, where $|x_k| = \max_{j=1\ldots,l+1} |x_j|$, and satisfies $\Phi_0(0) = 0$. Hence $\Phi_0(Q^+) = B^+$. Moreover, as shown in Section 2.3.1, Φ_0 is a uniform $\mathrm{d}\mu_\beta$-equivalence from Q^+ to B^+ in the sense of (2.3.1) and (2.3.2). Thus we have

$$\int_{Q^+} x_{l+1}^\beta |\nabla v|^2 \mathrm{d}\mathcal{H}^{l+1} \leq C \int_{B^+} x_{l+1}^\beta |\nabla v(\Phi_0^{-1}(x))|^2 \mathrm{d}\mathcal{H}^{l+1}(x). \tag{3.8.8}$$

Moreover, it follows from (3.8.4) that

$$|\nabla v(\Phi_0^{-1}(x))| = \frac{1}{|\Phi_0^{-1}(x)|}|\nabla v(\omega(\Phi_0^{-1}(x)))| \tag{3.8.9}$$

for every $x \in B^+$. By definition, $|\Phi_0^{-1}(x)| \geq C|x|$ for every $x \in B^+$ and a constant $C = C(m)$. For the remainder of this section C will always denote such a constant and we do not distinguish different C. Notice that $\omega(\Phi_0^{-1}(x)) = \omega(x)$ and hence, in view of (3.8.9), we deduce

$$|\nabla v(\Phi_0^{-1}(x))| \leq \frac{C}{|x|}|\nabla v(\omega(x))|. \tag{3.8.10}$$

It follows that

$$\int_{B^+} x_{l+1}^{\beta}|\nabla v(\Phi_0^{-1}(x))|^2 \mathrm{d}\mathcal{H}^{l+1}(x) \leq C\int_{B^+} \frac{1}{|x|^2} x_{l+1}^{\beta}|\nabla v(\omega(x))|^2 \mathrm{d}\mathcal{H}^{l+1}(x). \tag{3.8.11}$$

Now we change variables to r and ω. Recall the notation $\mathrm{d}\omega$, which we now use for the volume form with respect to the metric on $\partial^+B^+ = \mathbb{S}_+^{l+1}$ induced from the Euclidean metric on $\mathbb{R}^{l+1}$. We calculate

$$\begin{aligned}\int_{B^+} \frac{1}{|x|^2} x_{l+1}^{\beta}|\nabla v(\omega(x))|^2 \mathrm{d}\mathcal{H}^{l+1}(x) &= \int_0^1 \frac{1}{r^2}\int_{\mathbb{S}_+^l} r^{\beta}\omega_{l+1}^{\beta}|\nabla v(\omega)|^2 r^l \mathrm{d}\omega \mathrm{d}r \\ &= \int_0^1 r^{l-2+\beta}\int_{\mathbb{S}_+^l} \omega_{l+1}^{\beta}|\nabla v(\omega)|^2 \mathrm{d}\omega \mathrm{d}r \\ &= \frac{1}{l-1+\beta}\int_{\mathbb{S}_+^l} \omega_{l+1}^{\beta}|\nabla v|^2 \mathrm{d}\omega. \end{aligned} \tag{3.8.12}$$

Next we transform the right hand side of (3.8.12) into an integral over ∂^+Q^+, in terms of the gradient on ∂^+Q^+, via change of variables using the map Φ_0. Notice that $\Phi_0(x) = \omega(x)$ for all $x \in \partial^+Q^+$. Hence, in view of (3.8.4) we have

$$|\nabla v(\Phi_0(x))| = |\nabla v(\omega(x))| = |x||\nabla v(x)| \tag{3.8.13}$$

for all $x \in \partial^+Q^+$. For such x we also have $|x| \leq C$ and thus, using (3.8.13) and (3.8.7), we see that

$$|\nabla v(\Phi_0(x))| \leq C|\tilde{\nabla} v(x)| \tag{3.8.14}$$

for every $x \in \partial^+Q^+$, where we have now used $\tilde{\nabla}$ to denote the gradient on ∂^+Q^+. Hence, combining (3.8.14) with the fact that Φ_0 is a uniform $\mathrm{d}\mu_\beta$-equivalence between ∂^+Q^+ and ∂^+B^+, we change variables in (3.8.12). This yields

$$\begin{aligned}\int_{\partial^+B^+}\omega_{l+1}^{\beta}|\nabla v(\omega)|^2\mathrm{d}\omega &\leq C\int_{\partial^+Q^+}x_{l+1}^{\beta}|\nabla v(\Phi_0(x))|^2\mathrm{d}\mathcal{H}^l(x)\\ &\leq C\int_{\partial^+Q^+}x_{l+1}^{\beta}|\tilde{\nabla}v|^2\mathrm{d}\mathcal{H}^l. \qquad (3.8.15)\end{aligned}$$

Combining (3.8.8), (3.8.11), (3.8.12) and (3.8.15) we have

$$\begin{aligned}\int_{Q^+}x_{l+1}^{\beta}|\nabla v|^2\mathrm{d}\mathcal{H}^{l+1} &\leq C\int_{\partial^+Q^+}x_{l+1}^{\beta}|\tilde{\nabla}v|^2\mathrm{d}\mathcal{H}^l\\ &= C\sum_{\mathcal{F}^l}\int_{F^l}x_{l+1}^{\beta}|\tilde{\nabla}v|^2\mathrm{d}\mathcal{H}^l\\ &\quad + C\sum_{\mathcal{F}_{l+1}^l}\int_{F_{l+1}^l}x_{l+1}^{\beta}|\tilde{\nabla}v|^2\mathrm{d}\mathcal{H}^l, \qquad (3.8.16)\end{aligned}$$

where C depends on l and β. This is the estimate we seek for Q^+. We now rescale to Q_ρ^+ where Q_ρ has centre $y \in \partial\mathbb{R}_+^{l+1}$. Suppose $\hat{v}$ is homogeneous of degree zero with respect to such a y. Then $v(x) = \hat{v}(\rho x + y)$ is homogeneous of degree zero with respect to 0. We apply (3.8.16) to v on Q^+ to see that

$$\begin{aligned}\int_{Q_\rho^+}x_{l+1}^{\beta}|\nabla \hat{v}|^2\mathrm{d}\mathcal{H}^{l+1} &\leq C\rho\int_{\partial^+Q_\rho^+}x_{l+1}^{\beta}|\tilde{\nabla}\hat{v}|^2\mathrm{d}\mathcal{H}^l\\ &= C\rho\sum_{\mathcal{F}^l}\int_{F^l}x_{l+1}^{\beta}|\tilde{\nabla}\hat{v}|^2\mathrm{d}\mathcal{H}^l\\ &\quad + C\rho\sum_{\mathcal{F}_{l+1}^l}\int_{F_{l+1}^l}x_{l+1}^{\beta}|\tilde{\nabla}\hat{v}|^2\mathrm{d}\mathcal{H}^l, \qquad (3.8.17)\end{aligned}$$

where $\tilde{\nabla}$ is now the gradient on $\partial^+Q_\rho^+$.

A similar estimate holds on cubes $Q_\rho = \overline{Q_\rho^{l+1}(y)} \subset \mathbb{R}^{l+1}$ with the Euclidean metric. In this case, let F denote any l dimensional face of Q_ρ and write $\mathcal{F} = \mathcal{F}^l \cup \mathcal{F}_{l+1}^l$ for the collection of all such faces. If $\hat{v}$ is homogeneous of degree zero with respect to y then

$$\begin{aligned}\int_{Q_\rho}|\nabla \hat{v}|^2\mathrm{d}\mathcal{H}^{l+1} &\leq C\rho\int_{\partial Q_\rho}|\tilde{\nabla}\hat{v}|^2\mathrm{d}\mathcal{H}^l\\ &= C\rho\sum_{\mathcal{F}}\int_{F}|\tilde{\nabla}\hat{v}|^2\mathrm{d}\mathcal{H}^l, \qquad (3.8.18)\end{aligned}$$

which follows either from arguments analogous to those leading to (3.8.17) or directly from estimate (13) in the proof of the Luckhaus lemma as given in [46] section 2.12.2 after rescaling and translation.

Next we recall some properties of representations of functions in $W^{1,p}$.

3.8.2 Absolute Continuity Properties of Functions in $W_{\beta}^{1,2}$

We recall the discussion in [46] section 2.12.1 pertaining to some absolute continuity properties of $W^{1,p}$ functions. Let $\mathcal{H}^s$ denote the s-dimensional Hausdorff measure with respect to the Euclidean metric. Consider a rectangle $Q \subset \overline{\mathbb{R}_+^{m+1}}$ of the form $Q = [a_1, b_1] \times \ldots \times [a_{m+1}, b_{m+1}]$ where $a_i < b_i$. Suppose $v \in W_{\beta}^{1,2}(Q; \mathbb{R}^n)$ with $\beta \in (-1, 1)$. If $a_{m+1} > 0$ then $Q \subset \mathbb{R}_+^{m+1}$ and $v|_Q \in W^{1,2}(Q; \mathbb{R}^n)$ by Lemma 2.2.1.2. If $a_{m+1} = 0$ then $v|_Q \in W^{1,2}(Q; \mathbb{R}^n)$ if $\beta < 0$ by Lemma 2.2.1.3 and $v|_Q \in W^{1,p}(Q; \mathbb{R}^n)$ for $p \in (1, \frac{2}{1+\beta})$ if $\beta > 0$ by Lemma 2.2.1.4. Hence, by lemma 3.1.1 and theorem 3.1.8 in [32], if $a_{m+1} \geq 0$, we may infer the existence of a representative $\hat{v}$ of v such that, for each $i = 1, \ldots, m+1$, $\hat{v}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{m+1})$ is an absolutely continuous function of x_i for almost all fixed values of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+1}$ with respect to the m dimensional Hausdorff measure $\mathcal{H}^m$ on $[a_1, b_1] \times \ldots \times [a_{i-1}, b_{i-1}] \times [a_{i+1}, b_{i+1}] \times \ldots \times [a_{m+1}, b_{m+1}]$. The classical partial derivatives $\frac{\partial \hat{v}}{\partial x_i}$ agree almost everywhere with the weak derivatives $\frac{\partial v}{\partial x_i}$. Furthermore, for any closed subset N of $\mathbb{R}^n$, if $v(x) \in N$ for almost every x then it is possible to choose $\hat{v}(x) \in N$ for every $x \in \mathbb{R}_+^{m+1}$.

3.8.3 Embeddings of Absolutely Continuous Functions on Line Segments

A crucial aspect of the construction of the function w as described in Lemma 3.8.0.1, are bounds on the supremum of functions which are absolutely continuous along line segments parallel to the coordinate axis. More specifically, we are interested in bounds given in terms of the L^2 norm of these functions and their derivatives, either with respect to the Lebesgue measure or the Lebesgue measure with weight x_{m+1}^{β}.

Let E_j, for $j = 1, \ldots, m+1$, denote a 1 dimensional line segment of length $r \leq 1$, which is parallel to the x_j coordinate axis and satisfies $E_j \subset \mathbb{R}^m \times (0, 1]$. Let $v : \mathbb{R}_+^{m+1} \to \mathbb{R}^n$ be an absolutely continuous function on E_j and let $\partial_j v$ denote the weak partial derivative of v with respect to the jth variable for $j = 1, \ldots, m+1$.

Suppose v and $\partial_j v$ are square integrable on E_j. Integrating over E_j and using

Hölder's inequality we have

$$\sup_{E_j} |v|^2 \le \int_{E_j} |\partial_j |v|^2| \mathrm{d}x_j + \frac{1}{r} \int_{E_j} |v|^2 \mathrm{d}x_j$$
$$\le 2 \left(\int_{E_j} |\partial_j v|^2 \mathrm{d}x_j \right)^{\frac{1}{2}} \left(\int_{E_j} |v|^2 \mathrm{d}x_j \right)^{\frac{1}{2}} + \frac{1}{r} \int_{E_j} |v|^2 \mathrm{d}x_j \tag{3.8.19}$$

for $j = 1, \ldots, m+1$.

We will require estimates of in terms of $x_{m+1}^{\beta} \mathrm{d}x_j$ instead of the Lebesgue measure, where it is understood that for $j = 1, \ldots, m$ the factor x_{m+1}^{β} is evaluated on E_j. When $j = m+1$ we consider line segments depending on the sign of β. Suppose that E_{m+1} can be identified with $[a, b]$ where $0 \le a < b \le 1$ and $b - a = r$. In addition, assume v and $\partial_j v$ are square $x_{m+1}^{\beta} \mathrm{d}x_j$-integrable on E_j. Whenever $(\inf(x_{m+1}^{\beta}))^{-1}$ is finite, the required estimates follow from (3.8.19) for every $j = 1, \ldots, m+1$; if $\beta \in (-1, 0]$ and $0 \le a < b \le 1$ or if $\beta \in (0, 1)$ and $0 < a < b \le 1$ then v and $\partial_j v$ are square integrable on E_j and, in view of (3.8.19), we have

$$\sup_{E_j} |v|^2 \le \frac{2}{\inf_{E_j}(x_{m+1}^{\beta})} \left(\int_{E_j} x_{m+1}^{\beta} |\partial_j v|^2 \mathrm{d}x_j \right)^{\frac{1}{2}} \left(\int_{E_j} x_{m+1}^{\beta} |v|^2 \mathrm{d}x_j \right)^{\frac{1}{2}}$$
$$+ \frac{1}{\inf_{E_j}(x_{m+1}^{\beta}) r} \int_{E_j} x_{m+1}^{\beta} |v|^2 \mathrm{d}x_j. \tag{3.8.20}$$

Now let $\beta \in (0, 1)$ and consider $E_{m+1} = (0, r]$. In this case we can prove a similar but slightly weaker version of (3.8.20). Suppose that v and $\partial_{m+1} v$ are square $x_{m+1}^{\beta} \mathrm{d}x_{m+1}$-integrable. Then they are p-$\mathrm{d}x_{m+1}$-integrable for every $p \in (1, \frac{2}{1+\beta})$. We integrate along E_{m+1} to see that

$$\sup_{E_{m+1}} |v|^p \le \int_{E_{m+1}} |\partial_{m+1} |v|^p| \mathrm{d}x_{m+1} + \frac{1}{r} \int_{E_{m+1}} |v|^p \mathrm{d}x_{m+1}. \tag{3.8.21}$$

We consider the terms on the right hand side separately. It follows that

$$\int_{E_{m+1}} |\partial_{m+1} |v|^p| \mathrm{d}x_{m+1} \le p \int_{E_{m+1}} |\partial_{m+1} v| |v|^{p-1} \mathrm{d}x_{m+1}. \tag{3.8.22}$$

An application of Hölder's inequality yields

$$\begin{aligned}
&\int_{E_{m+1}} |\partial_{m+1} v||v|^{p-1} \mathrm{d}x_{m+1} \\
&= \int_{E_{m+1}} x_{m+1}^{-\frac{\beta}{2}} x_{m+1}^{\frac{\beta}{2}} |\partial_{m+1} v||v|^{p-1} \mathrm{d}x_{m+1} \\
&\leq \left(\int_{E_{m+1}} x_{m+1}^{\beta} |\partial_{m+1} v|^2 \mathrm{d}x_{m+1} \right)^{\frac{1}{2}} \left(\int_{E_{m+1}} x_{m+1}^{-\beta} |v|^{2(p-1)} \mathrm{d}x_{m+1} \right)^{\frac{1}{2}}. \qquad (3.8.23)
\end{aligned}$$

Then, since $\frac{1}{p-1}$ is greater than 1 (with conjugate exponent $q = \frac{1}{2-p}$), using Hölder's inequality again we see that

$$\begin{aligned}
&\int_{E_{m+1}} x_{m+1}^{-\beta} |v|^{2(p-1)} \mathrm{d}x_{m+1} \\
&= \int_{E_{m+1}} x_{m+1}^{-\beta} x_{m+1}^{-\beta(p-1)} x_{m+1}^{\beta(p-1)} |v|^{2(p-1)} \mathrm{d}x_{m+1} \\
&\leq \left(\int_{E_{m+1}} x_{m+1}^{-\frac{\beta p}{2-p}} \mathrm{d}x_{m+1} \right)^{2-p} \left(\int_{E_{m+1}} x_{m+1}^{\beta} |v|^2 \mathrm{d}x_{m+1} \right)^{p-1} \\
&= \left(\frac{2-p}{2-p-\beta p} \right)^{2-p} r^{2-p-\beta p} \left(\int_{E_{m+1}} x_{m+1}^{\beta} |v|^2 \mathrm{d}x_{m+1} \right)^{p-1} \qquad (3.8.24)
\end{aligned}$$

provided that $\beta < \frac{2}{p} - 1 = \frac{2-p}{p}$ or equivalently $p < \frac{2}{1+\beta}$ which is true by assumption. Combining (3.8.22), (3.8.23) and (3.8.24) gives

$$\begin{aligned}
&\int_{E_{m+1}} |\partial_{m+1} |v|^p| \mathrm{d}x_{m+1} \\
&\leq C r^{1-\frac{p}{2}-\frac{\beta p}{2}} \left(\int_{E_{m+1}} x_{m+1}^{\beta} |\partial_{m+1} v|^2 \mathrm{d}x_{m+1} \right)^{\frac{1}{2}} \left(\int_{E_{m+1}} x_{m+1}^{\beta} |v|^2 \mathrm{d}x_{m+1} \right)^{\frac{p-1}{2}} \qquad (3.8.25)
\end{aligned}$$

where $C = p \left(\frac{2-p}{2-p-\beta p} \right)^{1-\frac{p}{2}} = C(\beta)$ since p is chosen depending on β. Now we consider the other term in (3.8.21). Note that $\frac{2}{p} \geq 1$ with conjugate exponent $\frac{2}{2-p}$. Applying Hölder's inequality we see that

$$\begin{aligned}
\int_{E_{m+1}} |v|^p \mathrm{d}x_{m+1} &= \int_{E_{m+1}} x_{m+1}^{-\frac{\beta p}{2}} x_{m+1}^{\frac{\beta p}{2}} |v|^p \mathrm{d}x_{m+1} \\
&\leq \left(\int_{E_{m+1}} x_{m+1}^{-\frac{\beta p}{2-p}} \mathrm{d}x_{m+1} \right)^{\frac{2-p}{2}} \left(\int_{E_{m+1}} x_{m+1}^{\beta} |v|^2 \mathrm{d}x_{m+1} \right)^{\frac{p}{2}} \\
&= C r^{1-\frac{p}{2}-\frac{\beta p}{2}} \left(\int_{E_{m+1}} x_{m+1}^{\beta} |v|^2 \mathrm{d}x_{m+1} \right)^{\frac{p}{2}} \qquad (3.8.26)
\end{aligned}$$

where $C = C(\beta) = \left(\frac{2-p}{2-p-\beta p}\right)^{1-\frac{p}{2}}$. We combine (3.8.21) with (3.8.25) and (3.8.26) to see that

$$\begin{aligned}
&\sup_{E_{m+1}} |v|^p \\
&\le Cr^{1-\frac{p}{2}-\frac{\beta p}{2}} \left(\int_{E_{m+1}} x_{m+1}^{\beta} |\partial_{m+1} v|^2 \mathrm{d}x_{m+1}\right)^{\frac{1}{2}} \left(\int_{E_{m+1}} x_{m+1}^{\beta} |v|^2 \mathrm{d}x_{m+1}\right)^{\frac{p-1}{2}} \\
&\quad + Cr^{-\frac{p}{2}-\frac{\beta p}{2}} \left(\int_{E_{m+1}} x_{m+1}^{\beta} |v|^2 \mathrm{d}x_{m+1}\right)^{\frac{p}{2}}. \qquad (3.8.27)
\end{aligned}$$

In view of (3.8.27), we calculate

$$\begin{aligned}
&\sup_{E_{m+1}} |v|^2 \\
&= \left(\sup_{E_{m+1}} |v|^p\right)^{\frac{2}{p}} \\
&\le Cr^{\frac{2}{p}-1-\beta} \left(\int_{E_{m+1}} x_{m+1}^{\beta} |\partial_{m+1} v|^2 \mathrm{d}x_{m+1}\right)^{\frac{1}{p}} \left(\int_{E_{m+1}} x_{m+1}^{\beta} |v|^2 \mathrm{d}x_{m+1}\right)^{1-\frac{1}{p}} \\
&\quad + Cr^{-(1+\beta)} \int_{E_{m+1}} x_{m+1}^{\beta} |v|^2 \mathrm{d}x_{m+1}, \qquad (3.8.28)
\end{aligned}$$

where $C = C(\beta) = p^{\frac{2}{p}} \left(\frac{2-p}{2-p-\beta p}\right)^{\frac{2}{p}-1}$.

Remark 3.8.3.1. The estimates (3.8.20) and (3.8.28) are sufficient to prove the Sobolev embeddings $W_{\beta}^{1,2}(0,r) \hookrightarrow L^{\infty}[0,r]$ for $\beta \in (0,1)$ and $r \le 1$. Similarly (3.8.19) corresponds to the Sobolev embedding $W^{1,2}(a,b) \hookrightarrow L^{\infty}[a,b]$ for $a < b$.

3.8.4 Proof of Lemma 3.8.0.1

Proof of Lemma 3.8.0.1. We follow the proof, given in Section 2.12.2 of [46], of Lemma 1 in Section 2.6 of [46]. Throughout, C denotes a constant only depending on m and β. First we extend u and v to $\mathbb{R}^{m+1}$ in such a way that we can bound certain integrals of the extensions and their gradients on $Q_1^+(0)$ in terms of corresponding integrals involving u, v and their gradients on $\mathbb{S}_+^m$.

Suppose $u, v \in W_{\beta}^{1,2}(\mathbb{S}_+^m; N)$. We reflect u and v evenly in $\mathbb{R}^m \times \{0\}$, without relabelling, to get $u, v \in W_{\beta}^{1,2}(\mathbb{S}^m; N)$. We choose extensions of u and v to $\mathbb{R}^{m+1} \setminus \{0\}$ which are homogeneous of degree zero with respect to the origin. Then we choose representatives of these extensions which satisfy the absolute continuity properties described in Section 3.8.2 on $\overline{Q_1(0)}$. We will denote the representatives of the extensions of u and v by $\hat{u}$ and $\hat{v}$ respectively. Then $\hat{u}(\rho\omega) = \hat{u}(\omega)$,

$\hat{v}(\rho\omega) = \hat{v}(\omega)$ for almost every $\rho > 0$ and $\omega \in \mathbb{S}^m$. An application of (3.8.8), (3.8.11), (3.8.12) and (3.8.3) in Section 3.8.1 gives

$$\int_{Q_1^+(0)} x_{m+1}^{\beta} \left(|\nabla \hat{u}|^2 + |\nabla \hat{v}|^2\right) \mathrm{d}x \leq C \int_{\mathbb{S}_+^m} \omega_{m+1}^{\beta} \left(|Du|^2 + |Dv|^2\right) \mathrm{d}\omega, \qquad (3.8.29)$$

where D is the gradient on $\mathbb{S}_+^m$ and ∇ is the gradient on $\mathbb{R}_+^{m+1}$. In a similar way we find that

$$\int_{Q_1^+(0)} x_{m+1}^{\beta} |\hat{u} - \hat{v}|^2 \mathrm{d}x \leq C \int_{\mathbb{S}_+^m} \omega_{m+1}^{\beta} |u - v|^2 \mathrm{d}\omega. \qquad (3.8.30)$$

To construct w as required in the lemma, we first construct a map on $Q_{\frac{1}{4}}^+(0) \times [0, \varepsilon]$ whose gradient and distance from N can be bounded using combinations of the expressions on the right hand sides of (3.8.29) and (3.8.30). In order to achieve this we divide $Q_1(0)$ into small congruent sub-cubes of side length ε. We bound the sum, taken over all l-dimensional faces in this collection of cubes, of the integrals of a non-negative measurable function f over the faces in terms of the integral of f over $Q_1(0)$ for $l = 0, \ldots, m+1$. We will then apply these estimates to the functions $|\nabla \hat{u}| + |\nabla \hat{v}|^2$ and $|\hat{u} - \hat{v}|^2$.

More precisely, let $\varepsilon \in (0, \frac{1}{8})$ and define the closed rectangles $Q_{i,\varepsilon} = [i_1\varepsilon, (i_1 + 1)\varepsilon] \times \ldots \times [i_{m+1}\varepsilon, (i_{m+1} + 1)\varepsilon]$ for $i = (i_1, \ldots, i_{m+1}) \in \mathbb{Z}^{m+1}$. Fix $\varepsilon \in (0, \frac{1}{8})$ arbitrarily henceforth. Let F^l denote any l-dimensional face of a cube with no edges in the x_{m+1} direction and F^l_{m+1} any l-dimensional face with edges in the x_{m+1} direction. We define

$$\mathcal{Q} = \{Q_{i,\varepsilon} : i \in \mathbb{Z}^{m+1}, Q_{i,\varepsilon} \subset \overline{Q_{\frac{1}{2}}(0)}\},$$

$$\mathcal{F}_i^l = \left\{F^l \text{ faces of } Q_{i,\varepsilon}\right\},$$

and

$$\mathcal{F}_{i,m+1}^l = \left\{F_{m+1}^l \text{ faces of } Q_{i,\varepsilon}\right\}.$$

In addition, we write $x + \mathcal{F}_i^l$ to denote the collection of the translations of all faces in $\mathcal{F}_i^l$ by $x \in \mathbb{R}^{m+1}$ and $x + \mathcal{F}_{i,m+1}^l$ for the collection of the translations of all faces in $\mathcal{F}_{i,m+1}^l$ by $x \in \mathbb{R}^{m+1}$.

Consider a non-negative, measurable function $f : \overline{Q_1(0)} \to \mathbb{R}$ which is even with respect to the hyperplane $\mathbb{R}^m \times \{0\}$. Invoking [46] Section 2.12.2 estimate (3), which is a consequence of repeated applications of (2.3.35) for the Lebesgue measure and Fubini's theorem, we see that for every $K \geq 1$ there exists a set

$P \subset Q_{0,\varepsilon}$ of measure $|P| \leq \frac{C\varepsilon^{m+1}}{K}$, with $C = C(m)$, such that for all $y \in Q_{0,\varepsilon} \backslash P$ and $l \in \{0, \ldots, m+1\}$ we have

$$\begin{aligned} \varepsilon^{m+1-l} \sum_{\{i: Q_{i,\varepsilon} \in \mathcal{Q}\}} \left(\sum_{y+\mathcal{F}_i^l} \int_{F^l} f \mathrm{d}\mathcal{H}^l + \sum_{y+\mathcal{F}_{i,m+1}^l} \int_{F_{m+1}^l} f \mathrm{d}\mathcal{H}^l \right) &\leq K \int_{Q_1(0)} f \mathrm{d}x \\ &\leq 2K \int_{Q_1^+(0)} f \mathrm{d}x. \end{aligned} \tag{3.8.31}$$

Our aim is to eventually construct a function on $\mathbb{S}^m_+$ and hence we discard the integrals in (3.8.31) taken over any faces which do not intersect $\mathbb{R}^{m+1}_+$. In particular, since f is non negative, we find

$$\begin{aligned} &\varepsilon^{m+1-l} \sum_{\left\{ i: \substack{Q_{i,\varepsilon} \in \mathcal{Q} \\ i_{m+1} \geq -1} \right\}} \left(\sum_{y+\mathcal{F}_i^l} \int_{F^l \cap \mathbb{R}^{m+1}_+} f \mathrm{d}\mathcal{H}^l + \sum_{y+\mathcal{F}_{i,m+1}^l} \int_{F_{m+1}^l \cap \mathbb{R}^{m+1}_+} f \mathrm{d}\mathcal{H}^l \right) \\ &\leq 2K \int_{Q_1^+(0)} f \mathrm{d}x. \end{aligned} \tag{3.8.32}$$

We want to apply (3.8.32) to the squared norms of the gradients (on $\mathbb{R}^{m+1}$) of, and difference between, $\hat{u}$ and $\hat{v}$. Since we chose $\hat{u}$ and $\hat{v}$ with the absolute continuity properties described in Section 3.8.2 on $\overline{Q_1(0)}$ it follows that for almost every $x \in Q_{0,\varepsilon}$, with respect to the $m+1$-dimensional Lebesgue measure, all of the functions $\hat{u}, \hat{v}, \nabla \hat{u}, \nabla \hat{v}$ are $\mathcal{H}^l$ almost everywhere defined on each of the l-dimensional faces of $x + Q_{i,\varepsilon}$ for $Q_{i,\varepsilon} \in \mathcal{Q}$ and $l = 1, \ldots, m+1$. Moreover, the gradients of $\hat{u}$ and $\hat{v}$ on any l-dimensional face of $x + Q_{i,\varepsilon}$ coincide $\mathcal{H}^l$ almost everywhere with the tangential parts of $\nabla \hat{u}$ and $\nabla \hat{v}$ respectively. Thus we may choose $x = a \in Q_{0,\varepsilon}$ such that these properties hold and, provided we choose K (depending on m) sufficiently large in (3.8.32), such that $a_{m+1} \geq \frac{\varepsilon}{2}$ and such that we may apply (3.8.32) simultaneously for $f(x) = |x_{m+1}|^\beta \tilde{f}(x)$ with $\tilde{f}(x) = |\hat{u}(x) - \hat{v}(x)|^2$ and $\tilde{f}(x) = |\nabla \hat{u}(x)|^2 + |\nabla \hat{v}(x)|^2$ (where ∇ is the gradient on $\mathbb{R}^{m+1}$).

In particular, we have

$$
\begin{aligned}
&\varepsilon^{m+1-l} \sum_{\left\{i: \substack{Q_{i,\varepsilon}\in\mathcal{Q} \\ i_{m+1}\geq -1}\right\}} \sum_{a+\mathcal{F}_i^l} \int_{F^l\cap\mathbb{R}^{m+1}_+} x_{m+1}^{\beta} \tilde{f} \mathrm{d}\mathcal{H}^l \\
&+ \varepsilon^{m+1-l} \sum_{\left\{i: \substack{Q_{i,\varepsilon}\in\mathcal{Q} \\ i_{m+1}\geq -1}\right\}} \sum_{a+\mathcal{F}_{i,m+1}^l} \int_{F^l_{m+1}\cap\mathbb{R}^{m+1}_+} x_{m+1}^{\beta} \tilde{f} \mathrm{d}\mathcal{H}^l \\
&\leq C \int_{Q_1^+(0)} x_{m+1}^{\beta} \tilde{f} \mathrm{d}x. \qquad (3.8.33)
\end{aligned}
$$

Since $a_{m+1} \geq \frac{\varepsilon}{2}$ we assume hereafter that $a_{m+1} = c\varepsilon$ for $c \in \left[\frac{1}{2}, 1\right]$.

Now we begin the construction of w by defining a map on the one dimensional faces of every $Q \times [0, \varepsilon]$ where $Q = (a+Q_{i,\varepsilon}) \cap \mathbb{R}^{m+1}_+$ with $Q_{i,\varepsilon} \in \mathcal{Q}$ and $i_{m+1} \geq -1$. We show that the map we define satisfies bounds sufficient for us to apply (3.8.33).

Let E_j denote a one dimensional face of Q parallel to the jth coordinate axis for $j = 1, \ldots, m+1$. Define $w(x, 0) = \hat{u}(x)$ on $Q \times \{0\}$ and $w(x, \varepsilon) = \hat{v}(x)$ on $Q \times \{\varepsilon\}$. We extend w to $E_j \times [0, \varepsilon]$ by linear interpolation. That is, let $w(x, s) = (1 - \frac{s}{\varepsilon})\hat{u}(x) + \frac{s}{\varepsilon}\hat{v}(x)$ for $x \in E_j$ and $s \in [0, \varepsilon]$.

The distance of $w(x, s)$ from N can be bounded as follows. Since $\hat{u}(\mathbb{R}^{m+1}_+) \subset N$ by definition, it follows that

$$
\begin{aligned}
\mathrm{dist}^2(w(x,s), N) &= \left(\inf_{y\in N} |y - \left((1 - \frac{s}{\varepsilon})\hat{u}(x) + \frac{s}{\varepsilon}\hat{v}(x)\right)| \right)^2 \\
&\leq \left(\inf_{z\in Q_1^+(0)} |\hat{u}(z) - \hat{u}(x) + \frac{s}{\varepsilon}(\hat{u}(x) - \hat{v}(x))| \right)^2 \\
&\leq |\frac{s}{\varepsilon}(\hat{u}(x) - \hat{v}(x))|^2 \\
&\leq \max{}_{j=1,\ldots,m+1} \sup_{E_j} |\hat{u} - \hat{v}|^2 \qquad (3.8.34)
\end{aligned}
$$

for x in the 1-dimensional edges of Q and $s \in [0, \varepsilon]$. We give estimates for $\sup_{E_j} |\hat{u} - \hat{v}|^2$ by using the embeddings of absolutely continuous functions along line segments described in Section 3.8.3 which vary accordingly depending on whether $\inf_{E_j}(x_{m+1}^{\beta})$ is finite or not.

First we determine when $\inf_{E_j}(x_{m+1}^{\beta}) < \infty$. We see that this is satisfied on every E_j, with $j = 1, \ldots, m+1$, of any $Q = (a + Q_{i,\varepsilon}) \cap \mathbb{R}^{m+1}_+$ with $i_{m+1} \geq 0$. In this case, since $a_{m+1} \geq \frac{\varepsilon}{2}$, we have $\frac{\varepsilon}{2} \leq x_{m+1} \leq 1$ for x in any such Q. Hence, on any edge E_j, with $j = 1, \ldots, m+1$, of a Q with $i_{m+1} \geq 0$, if $\beta \in (-1, 0]$ then $\inf_{E_j}(x_{m+1}^{\beta}) \geq 1$, and if $\beta \in (0, 1)$ then $\inf_{E_j}(x_{m+1}^{\beta}) \geq \frac{\varepsilon^{\beta}}{2^{\beta}} \geq \frac{\varepsilon^{\beta}}{2}$. The same bounds hold on any E_j, with $j = 1, \ldots, m$, in a $Q = (a + Q_{i,\varepsilon}) \cap \mathbb{R}^{m+1}_+$ with $i_{m+1} = -1$ because such a cube has no edges in $\mathbb{R}^m \times \{0\}$ by definition. Furthermore, if $\beta \in (-1, 0]$ then on any such Q we still have $\inf_{E_{m+1}}(x^\beta_{m+1}) \geq 1$. Combining the previous considerations we deduce that, with the exception of the edges E_{m+1} belonging to a Q with $i_{m+1} = -1$ when $\beta \in (0, 1)$, $\inf_{E_j}(x^\beta_{m+1}) < \infty$ and satisfies the aforementioned bounds, for $j = 1, \ldots, m + 1$, on any edge E_j of any Q with $i_{m+1} \geq -1$. Hence it follows from (3.8.20), applied to the absolutely continuous function $\hat{u} - \hat{v}$, that on any edge E_j, with $j = 1, \ldots, m + 1$, of any $Q = (a + Q_{i,\varepsilon}) \cap \mathbb{R}^{m+1}_+$ with $i_{m+1} \geq -1$, if $\inf_{E_j}(x^\beta_{m+1}) < \infty$ then

$$\begin{aligned}
&\sup_{E_j} |\hat{u} - \hat{v}|^2 \\
&\leq \frac{C}{\varepsilon^{\frac{\beta}{2} + \frac{|\beta|}{2}}} \left(\int_{E_j} x^\beta_{m+1} \left(|\partial_j \hat{u}|^2 + |\partial_j \hat{v}|^2 \right) \mathrm{d}x_j \right)^{\frac{1}{2}} \left(\int_{E_j} x^\beta_{m+1} |\hat{u} - \hat{v}|^2 \mathrm{d}x_j \right)^{\frac{1}{2}} \\
&\quad + \frac{C}{\varepsilon^{1 + \frac{\beta}{2} + \frac{|\beta|}{2}}} \int_{E_j} x^\beta_{m+1} |\hat{u} - \hat{v}|^2 \mathrm{d}x_j. \qquad (3.8.35)
\end{aligned}$$

The remaining case we must consider occurs when $i_{m+1} = -1$ and $\beta \in (0, 1)$. Any Q which satisfies this assumption is of the form $Q = \overline{Q^m_{\frac{\varepsilon}{2}}(y)} \times (0, c\varepsilon]$ for some $y \in \mathbb{R}^m \times \{0\}$. This means that every E_{m+1} in Q has the form $\{x'\} \times (0, c\varepsilon]$ for appropriate $x' \in \overline{Q^m_{\frac{\varepsilon}{2}}(y)}$. We apply (3.8.28) to $\hat{u} - \hat{v}$ with a fixed $p \in (1, \frac{2}{1+\beta})$, noting that this assumption on p implies $(c\varepsilon)^{\frac{2}{p} - 1 - \beta} \leq 1$ and recalling that $c \geq 2^{-1}$, to see that

$$\begin{aligned}
&\sup_{E_{m+1}} |\hat{u} - \hat{v}|^2 \\
&\leq C \left(\int_{E_{m+1}} x^\beta_{m+1} |\partial_{m+1}(\hat{u} - \hat{v})|^2 \mathrm{d}x_{m+1} \right)^{\frac{1}{p}} \left(\int_{E_{m+1}} x^\beta_{m+1} |\hat{u} - \hat{v}|^2 \mathrm{d}x_{m+1} \right)^{1 - \frac{1}{p}} \\
&\quad + C\varepsilon^{-(1+\beta)} \int_{E_{m+1}} x^\beta_{m+1} |\hat{u} - \hat{v}|^2 \mathrm{d}x_{m+1}. \qquad (3.8.36)
\end{aligned}$$

The combination of (3.8.35) and (3.8.36) with (3.8.33), applied with $l = 1$, yields the following for any edge E_j of any $Q = a + Q_{i,\varepsilon} \cap \mathbb{R}^{m+1}_+$ with $Q_{i,\varepsilon} \in \mathcal{Q}$ and

$i_{m+1} \geq -1$. We see that

$$
\begin{aligned}
&\sup_{E_j} |\hat{u} - \hat{v}|^2 \\
&\leq \frac{C}{\varepsilon^{m+\frac{\beta}{2}+\frac{|\beta|}{2}}} \left(\int_{Q_1^+(0)} x_{m+1}^\beta \left(|\nabla \hat{u}|^2 + |\nabla \hat{v}|^2 \right) \mathrm{d}x \right)^{\frac{1}{q}} \left(\int_{Q_1^+(0)} x_{m+1}^\beta |\hat{u} - \hat{v}|^2 \mathrm{d}x \right)^{1-\frac{1}{q}} \\
&\quad + \frac{C}{\varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}} \int_{Q_1^+(0)} x_{m+1}^\beta |\hat{u} - \hat{v}|^2 \mathrm{d}x
\end{aligned} \tag{3.8.37}
$$

where $q \in \{p, 2\}$, for p fixed as above, depends on β.

We obtain a bound on $\operatorname{dist}^2(w(x,s), N)$, with $(x,s) \in E_j \times [0,\varepsilon]$ for any E_j of any $Q = a + Q_{i,\varepsilon} \cap \mathbb{R}_+^{m+1}$ with $Q_{i,\varepsilon} \in \mathcal{Q}$ and $i_{m+1} \geq -1$, by combining (3.8.34) and (3.8.37). In particular we find

$$
\begin{aligned}
&\operatorname{dist}^2(w(x,s), N) \\
&\leq \frac{C}{\varepsilon^{m+\frac{\beta}{2}+\frac{|\beta|}{2}}} \left(\int_{Q_1^+(0)} x_{m+1}^\beta \left(|\nabla \hat{u}|^2 + |\nabla \hat{v}|^2 \right) \mathrm{d}x \right)^{\frac{1}{q}} \left(\int_{Q_1^+(0)} x_{m+1}^\beta |\hat{u} - \hat{v}|^2 \mathrm{d}x \right)^{1-\frac{1}{q}} \\
&\quad + \frac{C}{\varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}} \int_{Q_1^+(0)} x_{m+1}^\beta |\hat{u} - \hat{v}|^2 \mathrm{d}x
\end{aligned} \tag{3.8.38}
$$

where q is as specified in (3.8.37).

Next we bound the gradient of w on the product of the 1-dimensional edges of Q with $[0,\varepsilon]$. Let $\overline{\nabla}$ denote the gradient on $E_j \times [0,\varepsilon]$ and $\tilde{\nabla}$ be the gradient on E_j for $j = 1, \ldots, m+1$. Recall that $\hat{u}, \hat{v}$ are defined so that the tangential parts of their gradients $\nabla \hat{u}, \nabla \hat{v}$ on $\mathbb{R}_+^{m+1}$ coincide $\mathcal{H}^1$ almost everywhere with their gradients $\tilde{\nabla} \hat{u}, \tilde{\nabla} \hat{v}$ on the edges E_j. Whenever $(x,s) \in E_j \times [0,\varepsilon]$ we calculate

$$
\begin{aligned}
\tilde{\nabla} w(x,s) &= (1 - \frac{s}{\varepsilon}) \tilde{\nabla} \hat{u}(x) + \frac{s}{\varepsilon} \tilde{\nabla} \hat{v}(x) \\
&= (1 - \frac{s}{\varepsilon}) \partial_j \hat{u}(x) + \frac{s}{\varepsilon} \partial_j \hat{v}(x)
\end{aligned}
$$

and

$$
\frac{\partial}{\partial s} w(x,s) = \frac{1}{\varepsilon} (\hat{v}(x) - \hat{u}(x)).
$$

It follows that

$$
\sup_{s \in [0,\varepsilon]} |\overline{\nabla} w(x,s)|^2 \leq 8 \left(|\nabla \hat{u}(x)|^2 + |\nabla \hat{v}(x)|^2 \right) + \frac{2}{\varepsilon^2} \left(|\hat{u}(x) - \hat{v}(x)|^2 \right), \tag{3.8.39}
$$

for x in any edge E_j, $j = 1, \ldots, m+1$, of Q and $s \in [0,\varepsilon]$. We integrate $|\overline{\nabla} w|^2$ over $E_j \times [0,\varepsilon]$ with respect to $x_{m+1}^\beta \mathrm{d}x_j \mathrm{d}s$ for $j = 1, \ldots, m+1$, evaluating x_{m+1}^β

at the fixed value of x_{m+1} which defines E_j as an edge in $Q \subset \mathbb{R}^{m+1}_+$ for $j \neq m+1$. In view of (3.8.39) we find

$$\int_{E_j \times [0,\varepsilon]} x_{m+1}^{\beta} |\overline{\nabla} w|^2 \mathrm{d}x_j \mathrm{d}s \leq 8\varepsilon \int_{E_j} x_{m+1}^{\beta} (|\nabla \hat{u}|^2 + |\nabla \hat{v}|^2) \mathrm{d}x_j + \frac{2}{\varepsilon} \int_{E_j} x_{m+1}^{\beta} |\hat{u} - \hat{v}|^2 \mathrm{d}x_j. \tag{3.8.40}$$

We now want to extend w to any $l+1$-dimensional ($l \geq 2$) face $F^l \times [0,\varepsilon]$ or $F^l_{m+1} \times [0,\varepsilon]$ of $Q \times [0,\varepsilon]$ in such a way as to preserve the estimate (3.8.38), regarding the distance of w from N, and so that we may apply (3.8.33) to obtain analogues of (3.8.40) with E_j replaced by F^l and F^l_{m+1}. The method for the extension of w is slightly different depending on whether $Q = (a + Q_{i,\varepsilon}) \cap \mathbb{R}^{m+1}_+$ is such that $i_{m+1} \geq 0$ or $i_{m+1} = -1$ but overall the idea is the same. In either case we proceed by induction; we assume w is defined on the l-dimensional faces of $Q \times [0,\varepsilon]$ and extend w homogeneously, of degree zero, into the $l+1$-dimensional faces.

Consider $Q \times [0,\varepsilon]$ for $Q = (a + Q_{i,\varepsilon}) \cap \mathbb{R}^{m+1}_+$ with $Q_{i,\varepsilon} \in \mathcal{Q}$ and $i_{m+1} \geq -1$. Suppose that $l \geq 2$ and w is already defined with L^2 gradient on every $F^{l-1} \times [0,\varepsilon]$ and square $x_{m+1}^{\beta} \mathrm{d}\mathcal{H}^{l-1}\mathrm{d}s$-integrable gradient on every $F^{l-1}_{m+1} \times [0,\varepsilon]$. In addition suppose that $w(x,0) = \hat{u}(x)$ and $w(x,\varepsilon) = \hat{v}(x)$ for $x \in F^l$ or $x \in F^l_{m+1}$. These assumptions imply that w is defined $\mathcal{H}^l$ almost everywhere on all the l-dimensional faces of Q for $l \geq 2$. Since $\partial(F^l \times [0,\varepsilon])$ and $\partial^+(F^l_{m+1} \times [0,\varepsilon])$ are the union of such l-dimensional faces, w is defined $\mathcal{H}^l$ almost everywhere on these sets. If Q is such that $i_{m+1} \geq 0$ then we extend w to each $F^l \times [0,\varepsilon]$ and $F^l_{m+1} \times [0,\varepsilon]$ by homogeneous extension of degree zero with respect to $(y, \frac{\varepsilon}{2})$, where y is the centre point of F^l or F^l_{m+1}. If $i_{m+1} = -1$ then we can extend w into $F^l \times [0,\varepsilon]$ using the same method. In this case we extend w homogeneously of degree 0 from $\partial^+(F^l_{m+1} \times [0,\varepsilon])$ into $F^l_{m+1} \times [0,\varepsilon]$ with respect to the point $\left(y^+, \frac{\varepsilon}{2}\right)$, where y is the centre point of F^l_{m+1} and $y^+ = y - (0, y_{m+1})$.

Now we check that this inductive construction of w preserves (3.8.38) and gives a bound analogous to (3.8.40) but with the E_j replaced by F^l and F^l_{m+1}. In what follows, $\overline{\nabla}$ denotes the gradient on the product spaces of the form $F^j \times [0,\varepsilon]$ and $F^j_{m+1} \times [0,\varepsilon]$ and $\tilde{\nabla}$ denotes the gradient on the faces F^j and F^j_{m+1} for $j = 2, \ldots, l$. Furthermore, unless stated otherwise, in the following inequalities $\overline{\nabla}$ and $\tilde{\nabla}$ are the gradients on the set over which they are integrated. We consider the cases $i_{m+1} \geq 0$ and $i_{m+1} = -1$ separately.

Suppose $i_{m+1} \geq 0$. Let F denote an l-dimensional face F^l or F^l_{m+1} of Q, let

y be the centre of F and let $\tilde{y} = \left(y, \frac{\varepsilon}{2}\right)$. Recall that $\hat{u}$ and $\hat{v}$ are chosen such that the tangential parts of the gradients $\nabla\hat{u}, \nabla\hat{v}$ on $\mathbb{R}^{m+1}$ coincide with $\tilde{\nabla}\hat{u}, \tilde{\nabla}\hat{v}$ for $\mathcal{H}^l$ almost every $x \in F$. We regard $F \times [0, \varepsilon]$ as an $l+1$ dimensional cube and combine the fact that $\sup_Q(x_{m+1}^\beta)\left(\inf_Q(x_{m+1}^\beta)\right)^{-1} \leq C$, a positive number independent of Q, ε and $\beta \in (-1, 1)$, with the estimate (3.8.18) from Section 3.8.1 to see that

$$\begin{aligned}
\int_{F\times[0,\varepsilon]} x_{m+1}^\beta |\overline{\nabla} w|^2 \mathrm{d}\mathcal{H}^l \mathrm{d}s \leq C\varepsilon \int_F x_{m+1}^\beta (|\nabla\hat{u}|^2 + |\nabla\hat{v}|^2)\mathrm{d}\mathcal{H}^l \\
+ C\varepsilon \sum_{a+\mathcal{F}_i^{l-1}} \int_{F^{l-1}\times[0,\varepsilon]} x_{m+1}^\beta |\overline{\nabla} w|^2 \mathrm{d}\mathcal{H}^{l-1}\mathrm{d}s \\
+ C\varepsilon \sum_{a+\mathcal{F}_{i,m+1}^{l-1}} \int_{F_{m+1}^{l-1}\times[0,\varepsilon]} x_{m+1}^\beta |\overline{\nabla} w|^2 \mathrm{d}\mathcal{H}^{l-1}\mathrm{d}s.
\end{aligned} \tag{3.8.41}$$

From (3.8.41), we inductively deduce that for any $l \in \{2, \ldots, m+1\}$ we can extend w to each $F^l \times [0, \varepsilon]$ and $F_{m+1}^l \times [0, \varepsilon]$ in $Q \times [0, \varepsilon]$ (with $i_{m+1} \geq 0$) so that w has an L^2 gradient $\overline{\nabla} w$ on these faces. Moreover, for any $\beta \in (-1, 1)$, $\overline{\nabla} w$ satisfies

$$\begin{aligned}
&\int_{F\times[0,\varepsilon]} x_{m+1}^\beta |\overline{\nabla} w|^2 \mathrm{d}\mathcal{H}^l \mathrm{d}s \\
&\leq C\varepsilon^{l-1} \sum_{a+\mathcal{F}_i^1} \int_{F^1\times[0,\varepsilon]} x_{m+1}^\beta |\overline{\nabla} w|^2 \mathrm{d}\mathcal{H}^1 \mathrm{d}s \\
&\quad + C\sum_{j=1}^l \varepsilon^{l-j+1} \sum_{a+\mathcal{F}_i^j} \int_{F^j} x_{m+1}^\beta (|\nabla\hat{u}|^2 + |\nabla\hat{v}|^2)\mathrm{d}\mathcal{H}^j \\
&\quad + C\varepsilon^{l-1} \sum_{a+\mathcal{F}_{i,m+1}^1} \int_{F_{m+1}^1\times[0,\varepsilon]} x_{m+1}^\beta |\overline{\nabla} w|^2 \mathrm{d}\mathcal{H}^1 \mathrm{d}s \\
&\quad + C\sum_{j=1}^l \varepsilon^{l-j+1} \sum_{a+\mathcal{F}_{i,m+1}^j} \int_{F_{m+1}^j} x_{m+1}^\beta (|\nabla\hat{u}|^2 + |\nabla\hat{v}|^2)\mathrm{d}\mathcal{H}^j.
\end{aligned} \tag{3.8.42}$$

Now we consider cubes $Q = (a + Q_{i,\varepsilon}) \cap \mathbb{R}_+^{m+1}$ with $i_{m+1} = -1$. We note that on any face $F^l \times [0, \varepsilon]$ of $Q \times [0, \varepsilon]$ we still have (3.8.42) because $\mathrm{dist}(F^l, \mathbb{R}^m \times \{0\}) \geq \frac{\varepsilon}{2}$, therefore we only consider the faces $F_{m+1}^l \times [0, \varepsilon]$.

Recall that we have assumed w is defined on $\partial^+(F_{m+1}^l \times [0, \varepsilon])$ and extended w homogeneously of degree zero into $F_{m+1}^l \times [0, \varepsilon]$ with respect to the point $(y^+, \frac{\varepsilon}{2})$, where $y = (y', y_{m+1})$ is the centre of F_{m+1}^l and $y^+ = (y', 0)$. We want to derive an estimate, analogous to (3.8.42), on these faces of Q. We do so by reducing the situation for $F^l_{m+1} \times [0,\varepsilon]$ to that of an $l+1$ dimensional half-cube and then applying (3.8.17) in Section 3.8.1.

Let $z = (z', z_{m+1}) \in \mathbb{R}^m \times (0,\infty)$ denote the centre of Q. We note that $Q = \overline{Q^m_{\frac{\varepsilon}{2}}(z^+)} \times (0, c\varepsilon]$ where $Q^m_{\frac{\varepsilon}{2}}(z^+) \subset \mathbb{R}^m \times \{0\}$, $z^+ = (z', 0)$ and $c \in \left[\frac{1}{2}, 1\right]$ is such that $a_{m+1} = c\varepsilon$. Define

$$\Psi_c : \overline{\mathbb{R}^{m+1}_+} \to \overline{\mathbb{R}^{m+1}_+} : x = (x', x_{m+1}) \mapsto \left(x', \frac{x_{m+1}}{2c}\right). \tag{3.8.43}$$

This is Lipschitz and C^1 with Lipschitz, C^1 inverse

$$\Psi_c^{-1} : \overline{\mathbb{R}^{m+1}_+} \to \overline{\mathbb{R}^{m+1}_+} : x = (x', x_{m+1}) \mapsto (x', 2cx_{m+1}). \tag{3.8.44}$$

Furthermore, we observe that $\Psi_c(Q) = \overline{Q^+_{\frac{\varepsilon}{2}}(z^+)} \cap \mathbb{R}^{m+1}_+$, a half-cube centred at z^+. We can write every l-dimensional face of $\Psi_c(Q)$ with edges parallel to the $m+1$-axis as $G^l_{m+1} = \Psi_c(F^l_{m+1})$ for appropriately corresponding faces F^l_{m+1} of Q. Notice that $\Psi_c(y^+) = y^+$, where $y^+ = (y', 0)$ and $y = (y', y_{m+1})$ is the centre of F^l_{m+1}. This will allow us to use w and Ψ_c to define a function $\hat{w}$ on $G^l_{m+1} \times [0,\varepsilon]$, which is homogeneous of degree zero with respect to $(y^+, \frac{\varepsilon}{2})$, and then apply (3.8.17).

Let $\hat{w}(x,s) = w(\Psi_c^{-1}(x), s)$ with domain $\Psi_c(Q) \times [0,\varepsilon]$. It follows that $\hat{w}$ is defined $\mathcal{H}^l$ almost everywhere on the l-dimensional faces of $\Psi_c(Q) \times [0,\varepsilon]$ since w is defined thus on the l-dimensional faces of $Q \times [0,\varepsilon]$. Furthermore, since w is homogeneous of degree zero in $F^l_{m+1} \times [0,\varepsilon]$ with respect to $\tilde{y} = \left(y^+, \frac{\varepsilon}{2}\right)$, it follows that $\hat{w}$ is homogeneous of degree 0 with respect to $\tilde{y}$ in $G^l_{m+1} \times [0,\varepsilon]$.

A calculation shows that Ψ_c is a uniform $\mathrm{d}\mu_\beta$ and x^β_{m+1}-equivalence from Q to $\Psi_c(Q) = \overline{Q_{\frac{\varepsilon}{2}}(p^+)} \cap \mathbb{R}^{m+1}_+$, as defined in Section 2.3.1. Since Ψ_c and Ψ_c^{-1} also have bounded derivatives, independently of $c \in [\frac{1}{2}, 1]$, using (2.3.2) from the aforementioned section, we calculate

$$\begin{aligned}
&\int_{F^l_{m+1} \times [0,\varepsilon]} x^\beta_{m+1} |\overline{\nabla} w(x,s)|^2 \mathrm{d}\mathcal{H}^l \mathrm{d}s \\
&\leq C \int_{G^l_{m+1} \times [0,\varepsilon]} x^\beta_{m+1} |\overline{\nabla} w(\Psi_c^{-1}(x), s)|^2 \mathrm{d}\mathcal{H}^l \mathrm{d}s \\
&\leq C \int_{G^l_{m+1} \times [0,\varepsilon]} x^\beta_{m+1} |\overline{\nabla} \hat{w}(x,s)|^2 \mathrm{d}\mathcal{H}^l \mathrm{d}s.
\end{aligned} \tag{3.8.45}$$

We identify $G^l_{m+1} \times [0,\varepsilon]$ with a half-cube $\overline{Q^{+,l+1}_{\frac{\varepsilon}{2}}(p)} \cap \mathbb{R}^{l+1}_+$ in such a way that $x_{m+1} \mapsto x_{l+1}$. This allows us to take advantage of the fact that $\hat{w}$ is homogeneous of degree zero with respect to $(y^+, \frac{\varepsilon}{2})$; we may apply (3.8.17) from Section 3.8.1 to see that

$$\int_{G^l_{m+1}\times[0,\varepsilon]} x^\beta_{m+1}|\overline{\nabla}\hat{w}|^2 \mathrm{d}\mathcal{H}^l \mathrm{d}s \leq C\varepsilon \int_{\partial^+(G^l_{m+1}\times[0,\varepsilon])} x^\beta_{m+1}|\overline{\nabla}\hat{w}|^2 \mathrm{d}\mathcal{H}^l, \tag{3.8.46}$$

where $\partial^+(\Omega\times[0,\varepsilon]) = \partial(\Omega\times[0,\varepsilon]) \cap (\mathbb{R}^{m+1}_+ \times [0,\varepsilon])$ for $\Omega \subset \mathbb{R}^{m+1}$. We transform the right hand side of (3.8.46) into an integral over $\partial^+(F^l_{m+1}\times[0,\varepsilon])$ using Ψ_c and (2.3.1) from Section 2.3.1. We have

$$\begin{aligned}\int_{\partial^+(G^l_{m+1}\times[0,\varepsilon])} x^\beta_{m+1}|\overline{\nabla}\hat{w}(x,s)|^2 \mathrm{d}\mathcal{H}^l &\leq C \int_{\partial^+(F^l_{m+1}\times[0,\varepsilon])} x^\beta_{m+1}|\overline{\nabla}\hat{w}(\Psi_c(x),s)|^2 \mathrm{d}\mathcal{H}^l \\ &\leq C \int_{\partial^+(F^l_{m+1}\times[0,\varepsilon])} x^\beta_{m+1}|\overline{\nabla}w(x,s)|^2 \mathrm{d}\mathcal{H}^l.\end{aligned} \tag{3.8.47}$$

We combine (3.8.45), (3.8.46) and (3.8.47) to deduce that

$$\begin{aligned}&\int_{F^l_{m+1}\times[0,\varepsilon]} x^\beta_{m+1}|\overline{\nabla}w|^2 \mathrm{d}\mathcal{H}^l \mathrm{d}s \\ &\leq C\varepsilon \int_{F^l_{m+1}} x^\beta_{m+1}(|\nabla\hat{u}|^2 + |\nabla\hat{v}|^2) \mathrm{d}\mathcal{H}^l \\ &\quad + C\varepsilon \sum_{a+\mathcal{F}^{l-1}_i} \int_{F^{l-1}\times[0,\varepsilon]} x^\beta_{m+1}|\overline{\nabla}w|^2 \mathrm{d}\mathcal{H}^{l-1}\mathrm{d}s \\ &\quad + C\varepsilon \sum_{a+\mathcal{F}^{l-1}_{i,m+1}} \int_{F^{l-1}_{m+1}\times[0,\varepsilon]} x^\beta_{m+1}|\overline{\nabla}w|^2 \mathrm{d}\mathcal{H}^{l-1}\mathrm{d}s.\end{aligned} \tag{3.8.48}$$

Thus, on any Q with $i_{m+1} = -1$, using (3.8.48) and (3.8.41), we inductively deduce that for any $l \in \{2,\dots,m+1\}$, we can extend w to each $F^l\times[0,\varepsilon]$ and $F^l_{m+1}\times[0,\varepsilon]$ in $Q\times[0,\varepsilon]$ so that w has an L^2 gradient $\overline{\nabla}w$ on $F^l\times[0,\varepsilon]$, and square $x_{m+1}^{\beta}\mathrm{d}\mathcal{H}^l\mathrm{d}s$-integrable gradient $\overline{\nabla}w$ on $F_{m+1}^l \times [0,\varepsilon]$. Furthermore, we have

$$
\begin{aligned}
&\int_{F_{m+1}^l \times [0,\varepsilon]} x_{m+1}^{\beta} |\overline{\nabla} w|^2 \mathrm{d}\mathcal{H}^l \mathrm{d}s \\
&\le C\varepsilon^{l-1} \sum_{a+\mathcal{F}_i^1} \int_{F^1 \times [0,\varepsilon]} x_{m+1}^{\beta} |\overline{\nabla} w|^2 \mathrm{d}\mathcal{H}^1 \mathrm{d}s \\
&\quad + C\sum_{j=1}^{l} \varepsilon^{l-j+1} \sum_{a+\mathcal{F}_i^j} \int_{F^j} x_{m+1}^{\beta} (|\nabla \hat{u}|^2 + |\nabla \hat{v}|^2) \mathrm{d}\mathcal{H}^j \\
&\quad + C\varepsilon^{l-1} \sum_{a+\mathcal{F}_{i,m+1}^1} \int_{F_{m+1}^1 \times [0,\varepsilon]} x_{m+1}^{\beta} |\overline{\nabla} w|^2 \mathrm{d}\mathcal{H}^1 \mathrm{d}s \\
&\quad + C\sum_{j=1}^{l} \varepsilon^{l-j+1} \sum_{a+\mathcal{F}_{i,m+1}^j} \int_{F_{m+1}^j} x_{m+1}^{\beta} (|\nabla \hat{u}|^2 + |\nabla \hat{v}|^2) \mathrm{d}\mathcal{H}^j. \qquad (3.8.49)
\end{aligned}
$$

So far, we have constructed a map $w = w^{i,\varepsilon}$ on each cube and rectangle $Q = (a+Q_{i,\varepsilon}) \cap \mathbb{R}_+^{m+1}$ such that $Q_{i,\varepsilon} \in \mathcal{Q}$ with $i_{m+1} \ge -1$. These maps satisfy the bound (3.8.38) on the distance from w to N on the products of one dimensional edges of Q with $[0,\varepsilon]$. Furthermore, for $1 \le l \le m+1$, (3.8.40) combined with (3.8.42) and (3.8.49) provide bounds for integrals of the norm of the gradients on the products of the l-dimensional faces of Q with $[0,\varepsilon]$, in terms of integrals of the norm of the gradients on the lower dimensional faces of Q. In particular, $w^{i,\varepsilon} \in W_{\beta}^{1,2}$ where it is defined. We now combine the definitions of $w^{i,\varepsilon}$ on each of the cubes Q with the aforementioned estimates in order to define the w as in the statement of the lemma.

It follows from the construction that $w^{(i,\varepsilon)} = w^{(j,\varepsilon)}$ $\mathcal{H}^{l+1}$-almost everywhere on common faces $F^l \times [0,\varepsilon]$ and $F_{m+1}^l \times [0,\varepsilon]$ of $(a+Q_{i,\varepsilon}) \cap \mathbb{R}_+^{m+1}$ and $(a+Q_{j,\varepsilon}) \cap \mathbb{R}_+^{m+1}$. Furthermore, for $0 < \varepsilon < \frac{1}{8}$ it follows that

$$
Q_{\frac{1}{4}}^{+}(0) \subset \bigcup_{\left\{i:\, \substack{Q_{i,\varepsilon} \in \mathcal{Q} \\ i_{m+1} \ge -1}\right\}} a + Q_{i,\varepsilon}.
$$

We may therefore define $w \in W_{\beta}^{1,2}(Q_{\frac{1}{4}}^{+}(0) \times [0,\varepsilon]; \mathbb{R}^n)$ by $w|_{(a+Q_{i,\varepsilon}) \cap \mathbb{R}_+^{m+1}}(x,s) = w^{(i,\varepsilon)}(x,s)$ for $s \in [0,\varepsilon]$.

Notice that since w is homogeneous of degree 0 on any l-dimensional face of any $Q \times [0,\varepsilon]$ with $l \ge 3$, our inductive procedure preserves (3.8.38) for all (x,s) in $Q_{\frac{1}{4}}^{+}(0) \times [0,\varepsilon]$, with the possible exception of a set P of m-dimensional Hausdorff measure 0. It follows from (3.8.38) that for $(x,s) \in (Q_{\frac{1}{4}}^{+}(0) \times [0,\varepsilon]) \backslash P$

we have

$$
\begin{aligned}
&\operatorname{dist}^2(w(x,s),N)\\
&\le \frac{C}{\varepsilon^{m+\frac{\beta}{2}+\frac{|\beta|}{2}}}\left(\int_{Q_1^+(0)} x_{m+1}^{\beta}\left(|\nabla\hat u|^2+|\nabla\hat v|^2\right)\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_{Q_1^+(0)} x_{m+1}^{\beta}|\hat u-\hat v|^2\mathrm{d}x\right)^{1-\frac{1}{q}}\\
&\quad+\frac{C}{\varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}}\int_{Q_1^+(0)} x_{m+1}^{\beta}|\hat u-\hat v|^2\mathrm{d}x
\end{aligned}\tag{3.8.50}
$$

where q is as specified in (3.8.37). Moreover, we combine (3.8.40) with (3.8.42) and (3.8.49), which we apply with $l=m+1$, and apply (3.8.33) for $l=1,\dots,m+1$, to see that

$$
\begin{aligned}
\int_{Q^+_{\frac{1}{4}}(0)\times[0,\varepsilon]} x_{m+1}^{\beta}|\overline{\nabla}w|^2\mathrm{d}x\mathrm{d}s \le C\varepsilon\int_{Q_1^+(0)} x_{m+1}^{\beta}\left(|\nabla\hat u|^2+|\nabla\hat v|^2\right)\mathrm{d}x&\\
+\frac{C}{\varepsilon}\int_{Q_1^+(0)} x_{m+1}^{\beta}|\hat u-\hat v|^2\mathrm{d}x.&
\end{aligned}\tag{3.8.51}
$$

The definition of w as required now follows from combining (3.8.50) and (3.8.51) with (3.8.29) and (3.8.30). The absolute continuity properties, described in Section 3.8.2, of w, viewed as a function defined on a rectangle in polar coordinates, guarantee that for almost every $\rho\in[\frac{1}{8},\frac{1}{4}]$, w has square $x_{m+1}^{\beta}\mathrm{d}\mathcal{H}^m\mathrm{d}s$-integrable gradient $\partial^+B_\rho^+(0)\times[0,\varepsilon]$ which coincides $\mathcal{H}^m\mathrm{d}s$ almost everywhere with the tangential part of $\overline{\nabla}w$. Using Fubini's theorem and (2.3.35) from Section 2.3.4, applied to the map $\rho\mapsto\int_{\partial^+B_\rho^+(0)\times[0,\varepsilon]} x_{m+1}^{\beta}|\overline{\nabla}w|^2\mathrm{d}\mathcal{H}^m\mathrm{d}s$, we may therefore choose $\rho\in[\frac{1}{8},\frac{1}{4}]$ such that w has square $x_{m+1}^{\beta}\mathrm{d}\mathcal{H}^m\mathrm{d}s$-integrable gradient on $\partial^+B_\rho^+(0)\times[0,\varepsilon]$ and satisfies

$$
\int_{\partial^+B_\rho^+(0)\times[0,\varepsilon]} x_{m+1}^{\beta}|\overline{\nabla}w|^2\mathrm{d}\mathcal{H}^m\mathrm{d}s\le C\int_{B^+_{\frac{1}{4}}(0)\times[0,\varepsilon]} x_{m+1}^{\beta}|\overline{\nabla}w|^2\mathrm{d}\mathcal{H}^{m+1}\mathrm{d}s.\tag{3.8.52}
$$

We define $\tilde w$ on $\mathbb{S}^m_+\times[0,\varepsilon]$ by $\tilde w(\omega,s)=w(\rho\omega,s)$. It follows that $\tilde w(\omega,0)=\hat u(\omega)$ almost everywhere on $\mathbb{S}^m_+\times\{0\}$ and $\tilde w(\omega,\varepsilon)=\hat v(\omega)$ almost everywhere on $\mathbb{S}^m\times\{\varepsilon\}$ because the corresponding properties hold for w on $\partial^+B_\rho^+(0)\times[0,\varepsilon]$. Furthermore, since $\rho\in[\frac{1}{8},\frac{1}{4}]$, the map $\omega\to\rho\omega$ is a uniform $\mathrm{d}\mu_\beta$ and x_{m+1}^{β}-equivalence as described in Section 2.3.1 and thus

$$
\int_{\mathbb{S}^m_+\times[0,\varepsilon]} x_{m+1}^{\beta}|\overline{D}\hat w|^2\mathrm{d}\omega\mathrm{d}s\le C\int_{\partial^+B_\rho^+(0)\times[0,\varepsilon]} x_{m+1}^{\beta}|\overline{\nabla}w|^2\mathrm{d}\mathcal{H}^m\mathrm{d}s\tag{3.8.53}
$$

where $\overline{D}$ is the gradient on $\mathbb{S}^m_+\times[0,\varepsilon]$. We combine (3.8.51), (3.8.52) and (3.8.53) with (3.8.29) and (3.8.30) to give (3.8.1). Moreover, $\hat{w}(\omega, s) = w(\rho\omega, s)$ for $\mathcal{H}^m \mathrm{d}s$ almost every $(\omega, s) \in \mathbb{S}^m_+ \times [0, \varepsilon]$ and therefore $\hat{w}$ satisfies (3.8.50), since the set of points P for which this statement fails for w has vanishing m-dimensional Hausdorff measure. Together, (3.8.50), (3.8.29) and (3.8.30) yield (3.8.2). This concludes the proof. $\square$

3.9 Corollary to the Luckhaus Lemma

3.9.1 Radial Slicing for Functions in $W^{1,2}_\beta$

Before stating the corollary to Lemma 3.8.0.1 we discuss the relationship between integrals of a $W^{1,2}_\beta$ function on $B^+_\rho(y)$ and integrals of its restriction to $\partial^+ B^+_\sigma(y)$ for $\sigma \leq \rho$, similarly to the discussion in section 2.7 of [46].

For $v \in W^{1,2}_\beta(B^+_\rho(y); N)$ and $\sigma \in (\frac{\rho}{2}, \rho)$ we consider $\hat{v}(\omega) := v(y + \sigma\omega)$ where $\omega \in \mathbb{S}^m_+$. Let D denote the gradient on $\mathbb{S}^m_+$ and ∇ the gradient on $\mathbb{R}^{m+1}_+$. Using (2.3.35) from the discussion in Section 2.3.4, if $\theta \in (0, 1)$ then for all $\sigma \in (\frac{\rho}{2}, \rho)$ with the exception of a set of 1-dimensional Lebesgue measure $\frac{\theta\rho}{2}$, we find

$$\begin{aligned}
\int_{\mathbb{S}^m_+} \omega_{m+1}^\beta |D\hat{v}|^2 \mathrm{d}\omega &= \int_{\mathbb{S}^m_+} \omega_{m+1}^\beta |D(v(y + \sigma\omega))|^2 \mathrm{d}\omega \\
&\leq C\sigma^{2-m-\beta} \int_{\mathbb{S}^m_+} (\sigma\omega_{m+1})^\beta |\nabla v(y + \sigma\omega)|^2 \sigma^m \mathrm{d}\omega \\
&= C\sigma^{2-m-\beta} \int_{\partial^+ B^+_\sigma(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}S(x) \\
&\leq \frac{C}{\theta} \rho^{1-m-\beta} \int_{B^+_\rho(y) \setminus B^+_{\frac{\rho}{2}}(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x
\end{aligned} \tag{3.9.1}$$

where $\mathrm{d}S(x)$ is the Euclidean volume element on $\partial^+ B^+_\sigma(y)$. Similarly, for all $\sigma \in (\frac{\rho}{2}, \rho)$ with the exception of a set of 1-dimensional Lebesgue measure $\frac{\theta\rho}{2}$ we have

$$\begin{aligned}
\int_{\mathbb{S}^m_+} \omega_{m+1}^\beta |\hat{v} - \overline{v}_{B^+_\rho(y),\beta}|^2 \mathrm{d}\omega \leq& \sigma^{-m-\beta} \int_{\partial^+ B^+_\sigma(y)} x_{m+1}^\beta |v - \overline{v}_{B^+_\rho(y),\beta}|^2 \mathrm{d}S(x) \\
&\leq \frac{C}{\theta} \rho^{-(1+m+\beta)} \int_{B^+_\rho(y) \setminus B^+_{\frac{\rho}{2}}(y)} x_{m+1}^\beta |v - \overline{v}_{B^+_\rho(y),\beta}|^2 \mathrm{d}x.
\end{aligned} \tag{3.9.2}$$

Hence if we choose $\theta \in (0, 1)$ small enough, depending on at most m, β, in (3.9.1) and (3.9.2) we may choose a $\sigma \in (\frac{3\rho}{4}, \rho)$ such that they hold simultaneously and so that $\hat{v} \in W^{1,2}_{\beta}(\mathbb{S}^m_+; N)$.

Now we state a corollary to Lemma 3.8.0.1 which gives the estimate (3.7.1). The following corollary is an analogue of Corollary 1 in Section 2.7.

3.9.2 Luckhaus Corollary

Corollary 3.9.2.1. *There exists a $\delta_0 = \delta_0(m, N, \beta) > 0$ such that the following holds. Let $\varepsilon \in (0,1)$ and $v \in W^{1,2}_{\beta}(B^+_\rho(y); N)$ with $\rho^{1-m-\beta} \int_{B^+_\rho(y)} x^{\beta}_{m+1} |\nabla v|^2 \mathrm{d}x \leq \delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}$. Then there is a $\sigma \in (\frac{3\rho}{4}, \rho)$ such that we can find a map $w_\varepsilon \in W^{1,2}_{\beta}(B^+_\rho(y); N)$ which agrees with v in $B^+_\rho(y) \backslash B^+_\sigma(y)$ and such that*

$$\begin{aligned} &\sigma^{1-m-\beta} \int_{B^+_\sigma(y)} x^{\beta}_{m+1} |\nabla w|^2 \mathrm{d}x \\ &\leq C\varepsilon \rho^{1-m-\beta} \int_{B^+_\rho(y)} x^{\beta}_{m+1} |\nabla v|^2 \mathrm{d}x + \frac{C}{\varepsilon} \rho^{-(1+m+\beta)} \int_{B^+_\rho(y)} x^{\beta}_{m+1} |v - \overline{v}_{B^+_\rho(y),\beta}|^2 \mathrm{d}x \end{aligned} \tag{3.9.3}$$

for a constant $C = C(m, \beta)$.

Proof. We follow the proof of Corollary 1 in Section 2.7 of [46]. Our strategy is the following. We find a $\lambda \in N$ which can be chosen close to $\overline{v}_{B^+_\rho(y),\beta}$, provided the energy of v on $B^+_\rho(y)$ is sufficiently small. Then we choose a $\sigma \in (\frac{3\rho}{4}, \rho)$ which allows us to control certain integrals involving $v|_{\partial^+ B^+_\sigma(y)}$ in terms of the energy of v on $B^+_\rho(y)$. Using Lemma 3.8.0.1 we obtain a w_0 agreeing with $v|_{\partial^+ B^+_\sigma(y)}$ on $\mathbb{S}^m_+ \times \{0\}$ and λ on $\mathbb{S}^m_+ \times \{\varepsilon\}$ in the sense of traces. We then show that we can make the distance from w_0 to N small enough so that we may project w_0 onto N using the nearest point projection. Using the resulting map we interpolate between v and λ to define a map which satisfies the statement of the lemma.

Throughout, C denotes a constant which depends on m and possibly β and we only distinguish different C when necessary. We will also assume that $\varepsilon \leq \frac{1}{2}$. This is a technical assumption and we obtain the lemma for $\varepsilon \in (\frac{1}{2}, 1)$ by applying the lemma for $\varepsilon = \frac{1}{2}$, after shrinking the δ_0 we obtained for $\varepsilon \leq \frac{1}{2}$ by a factor depending only on m, β if necessary.

Let $\delta_0 > 0$ to be chosen as required and suppose the assumptions of the lemma hold for δ_0. First, we bound the distance of $\overline{v}_{B^+_\rho(y),\beta}$ from N in terms of δ_0 and ε.

An application of the Poincaré Inequality, Lemma 2.3.3.3, gives

$$\begin{aligned}\rho^{-(1+m+\beta)} \int_{B_\rho^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 \mathrm{d}x &\leq C\rho^{1-m-\beta} \int_{B_\rho^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \\ &\leq C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}. \end{aligned} \tag{3.9.4}$$

As discussed in Section 3.8.2, we may work with a representative of v, which we don't relabel, such that $v(B_\rho^+(y)) \subset N$. It follows that

$$\mathrm{dist}^2(\overline{v}_{B_\rho^+(y),\beta}, N) \leq |v(x) - \overline{v}_{B_\rho^+(y),\beta}|^2 \tag{3.9.5}$$

for every $x \in B_\rho^+(y)$. Now notice that we can choose a constant C such that $C\rho^{-(1+m+\frac{\beta}{2}+\frac{|\beta|}{2})} \geq \left(\int_{B_\rho^+(y)} x_{m+1}^{\frac{\beta}{2}+\frac{|\beta|}{2}} \mathrm{d}x \right)^{-1}$. Therefore, integrating (3.9.5) over $B_\rho^+(y)$ with respect to $x_{m+1}^{\frac{\beta}{2}+\frac{|\beta|}{2}} \mathrm{d}x$ and dividing by $\int_{B_\rho^+(y)} x_{m+1}^{\frac{\beta}{2}+\frac{|\beta|}{2}} \mathrm{d}x$ gives

$$\mathrm{dist}^2(\overline{v}_{B_\rho^+(y),\beta}, N) \leq C\rho^{-(1+m+\frac{\beta}{2}+\frac{|\beta|}{2})} \int_{B_\rho^+(y)} x_{m+1}^{\frac{\beta}{2}+\frac{|\beta|}{2}} |v - \overline{v}_{B_\rho^+(y),\beta}|^2 \mathrm{d}x. \tag{3.9.6}$$

If $\beta \in (-1, 0)$ we note additionally that $\rho^\beta \leq \inf_{B_\rho^+(y)} x_{m+1}^\beta$ and thus, in view of (3.9.6), we conclude that

$$\mathrm{dist}^2(\overline{v}_{B_\rho^+(y),\beta}, N) \leq C\rho^{-(1+m+\beta)} \int_{B_\rho^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 \mathrm{d}x \tag{3.9.7}$$

for any $\beta \in (-1, 1)$. Combining (3.9.7) with (3.9.4) we find

$$\mathrm{dist}^2(\overline{v}_{B_\rho^+(y),\beta}, N) \leq C\rho^{-(1+m+\beta)} \int_{B_\rho^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 \mathrm{d}x \leq C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}. \tag{3.9.8}$$

As a consequence, we may choose $\lambda \in N$ such that

$$|\lambda - \overline{v}_{B_\rho^+(y),\beta}|^2 \leq C\rho^{-(1+m+\beta)} \int_{B_\rho^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_\rho^+(y),\beta}|^2 \mathrm{d}x \leq C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}. \tag{3.9.9}$$

We want to apply Lemma 3.8.0.1 to v and λ. Since $\lambda \in N$ is constant and $v \in N$ by construction, we are permitted to do this provided we can use v to define a function in $W_\beta^{1,2}(\mathbb{S}_+^m; N)$. Let $\omega \in \mathbb{S}_+^m$. The combination of (3.9.4) with (3.9.1) and (3.9.2) from Section 3.9.1 yields a $\sigma \in (\frac{3\rho}{4}, \rho)$ such that $\hat{v} \in W_\beta^{1,2}(\mathbb{S}_+^m; N)$, where $\hat{v}(\omega) = v(\sigma\omega + y)$ and, moreover, such that

$$\begin{aligned}\int_{\mathbb{S}^m_+} \omega_{m+1}^{\beta} |D\hat{v}|^2 \mathrm{d}\omega &\le C\rho^{1-m-\beta} \int_{B^+_\rho(y)\setminus B^+_{\frac{\rho}{2}}(y)} x^{\beta}_{m+1} |\nabla v|^2 \mathrm{d}x \\ &\le C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}, \end{aligned} \tag{3.9.10}$$

where D is the gradient on $\mathbb{S}^m_+$, and

$$\begin{aligned}\int_{\mathbb{S}^m_+} \omega_{m+1}^{\beta} |\hat{v} - \overline{v}_{B^+_\rho(y),\beta}|^2 \mathrm{d}\omega &\le C\rho^{-(1+m+\beta)} \int_{B^+_\rho(y)\setminus B^+_{\frac{\rho}{2}}(y)} x^{\beta}_{m+1} |v - \overline{v}_{B^+_\rho(y),\beta}|^2 \mathrm{d}x \\ &\le C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}. \end{aligned} \tag{3.9.11}$$

Thus we may apply Lemma 3.8.0.1 to $\hat{v} \in W^{1,2}_\beta(\mathbb{S}^m_+; N)$ and λ. This yields a $w_0 : \mathbb{S}^m_+ \times [0,\varepsilon] \to \mathbb{R}^n$ with $w_0 = \hat{v}$ on $\mathbb{S}^m_+ \times \{0\}$ and $w_0 = \lambda$ on $\mathbb{S}^m_+ \times \{\varepsilon\}$ in the sense of traces. Furthermore, in view of (3.8.1) we have

$$\int_{\mathbb{S}^m_+\times[0,\varepsilon]} \omega_{m+1}^{\beta} |\overline{D} w_0|^2 \mathrm{d}\omega \mathrm{d}s \le C\varepsilon \int_{\mathbb{S}^m_+} \omega_{m+1}^{\beta} |D\hat{v}|^2 \mathrm{d}\omega + \frac{C}{\varepsilon} \int_{\mathbb{S}^m_+} \omega_{m+1}^{\beta} |\hat{v} - \lambda|^2 \mathrm{d}\omega, \tag{3.9.12}$$

where $\overline{D}$ is the gradient on $\mathbb{S}^m_+ \times [0,\varepsilon]$ and D is the gradient on $\mathbb{S}^m_+$. In addition, it follows from (3.8.2) that

$$\begin{aligned}\mathrm{dist}^2(w_0(\omega,s), N) \le{}& \frac{C}{\varepsilon^{m+\frac{\beta}{2}+\frac{|\beta|}{2}}} \left(\int_{\mathbb{S}^m_+} \omega_{m+1}^{\beta} |D\hat{v}|^2 \mathrm{d}\omega \right)^{\frac{1}{q}} \left(\int_{\mathbb{S}^m_+} \omega_{m+1}^{\beta} |\hat{v} - \lambda|^2 \mathrm{d}\omega \right)^{1-\frac{1}{q}} \\ &+ \frac{C}{\varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}} \int_{\mathbb{S}^m_+} \omega_{m+1}^{\beta} |\hat{v} - \lambda|^2 \mathrm{d}\omega \end{aligned} \tag{3.9.13}$$

for every $(\omega, s) \in \mathbb{S}^m_+ \times [0,\varepsilon]$, where $q \in (1,2]$ depends on β. We proceed to bound $\mathrm{dist}(w_0(\omega,s), N)$ in terms of δ_0. Henceforth we assume that $\delta_0 \le 1$. Using (3.9.9) and (3.9.11) we deduce that

$$\begin{aligned}\int_{\mathbb{S}^m_+} \omega_{m+1}^{\beta} |\hat{v} - \lambda|^2 \mathrm{d}\omega &\le 2\int_{\mathbb{S}^m_+} \omega_{m+1}^{\beta} |\hat{v} - \overline{v}_{B^+_\rho(y),\beta}|^2 \mathrm{d}\omega \\ &\quad + 2\int_{\mathbb{S}^m_+} \omega_{m+1}^{\beta} |\overline{v}_{B^+_\rho(y),\beta} - \lambda|^2 \mathrm{d}\omega \\ &\le 2\int_{\mathbb{S}^m_+} \omega_{m+1}^{\beta} |\hat{v} - \overline{v}_{B^+_\rho(y),\beta}|^2 \mathrm{d}\omega + C|\overline{v}_{B^+_\rho(y),\beta} - \lambda|^2 \\ &\le C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}} + C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}} \\ &\le C\delta_0^2 \varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}. \end{aligned} \tag{3.9.14}$$

The combination of (3.9.13) with (3.9.10) and (3.9.14) yields

$$\begin{aligned}\operatorname{dist}^2(w_0(\omega,s),N) \leq{}& \frac{C}{\varepsilon^{m+\frac{\beta}{2}+\frac{|\beta|}{2}}}\left(C\delta_0^2\varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}\right)^{\frac{1}{q}}\left(C\delta_0^2\varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}\right)^{1-\frac{1}{q}}\\ &+\frac{C}{\varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}}}C\delta_0^2\varepsilon^{m+1+\frac{\beta}{2}+\frac{|\beta|}{2}} \qquad (3.9.15)\end{aligned}$$

for every $(\omega,s)\in\mathbb{S}^m_+\times[0,\varepsilon]$ and for $q\in(1,2]$ depending on β. Hence, using (3.9.15), we find

$$\operatorname{dist}(w_0(x,s),N)\leq C\delta_0. \qquad (3.9.16)$$

In order to construct w as required we choose δ_0 small enough to allow us to apply the nearest point projection of N to w_0. In particular, we choose δ_0, depending on N,m,β, such that $C\delta_0\leq\hat{\alpha}$ where C is the constant in (3.9.16) and $\hat{\alpha}>0$ is sufficiently small to guarantee that the nearest point projection onto N, which we denote π_N, exists and has bounded derivatives in $N_{\hat{\alpha}}=\{x\in\mathbb{R}^n:\operatorname{dist}(x,N)\leq\hat{\alpha}\}$. It then follows from (3.9.16) that we may apply π_N to w_0. Let $\omega\in\mathbb{S}^m_+$ satisfy $\omega=\omega(x)=\frac{x-y}{|x-y|}$, $r=|x-y|$ and define $w\in W^{1,2}_\beta(B^+_\rho(y);N)$ by

$$w(x)=w(y+r\omega(x))=\begin{cases} v(y+r\omega(x)) & r\in(\sigma,\rho)\\ \pi_N(w_0(\omega(x),(1-\frac{r}{\sigma}))) & r\in[(1-\varepsilon)\sigma,\sigma]\\ \lambda & r\in(0,(1-\varepsilon)\sigma).\end{cases}$$

Note that w agrees with v in $B^+_\rho(y)\backslash B^+_\sigma(y)$. To complete the proof we check that w satisfies (3.9.3). By definition, $\nabla w=0$ in $B^+_{(1-\varepsilon)\sigma}(y)$ which implies

$$\sigma^{1-m-\beta}\int_{B^+_\sigma(y)}x^\beta_{m+1}|\nabla w|^2\mathrm{d}x=\sigma^{1-m-\beta}\int_{B^+_\sigma(y)\backslash B^+_{(1-\varepsilon)\sigma}(y)}x^\beta_{m+1}|\nabla w|^2\mathrm{d}x. \qquad (3.9.17)$$

Observe that $\nabla w(x)=\nabla(\pi_N(w_0(\omega(x),1-\sigma^{-1}r(x))))$ in $B^+_\sigma(y)\backslash B^+_{(1-\varepsilon)\sigma}(y)$. Using the chain rule we deduce that

$$\begin{aligned}&|\nabla(\pi_N(w_0(\omega,1-\sigma^{-1}r)))|\\ &\leq C|\nabla\pi_N(w_0(\omega,1-\sigma^{-1}r))||\overline{D}w_0(\omega,1-\sigma^{-1}r)||\nabla(\omega,1-\sigma^{-1}r)|,\end{aligned}$$

where $\overline{D}$ is the gradient on $\mathbb{S}^m_+\times[0,\varepsilon]$. The boundedness of the derivatives of π_N on $N_{\hat{\alpha}}$ implies $|\nabla\pi_N(w_0(\omega,1-\sigma^{-1}r))|\leq C$. Recall the technical assumption $\varepsilon\leq\frac{1}{2}$, together with the fact that $r\in[(1-\varepsilon)\sigma,\sigma]$, $\omega(x)=\frac{x-y}{|x-y|}$ and $r(x)=|x-y|$. Using the preceding facts we calculate $|\nabla\omega|\leq Cr^{-1}\leq C((1-\varepsilon)\sigma)^{-1}\leq C\sigma^{-1}$

and $|\nabla(1-\sigma^{-1}r)| \leq C\sigma^{-1}$ which yields $|\nabla(\omega, 1-\sigma^{-1}r)| \leq C\sigma^{-1}$. It follows that

$$|\nabla(\pi_N(w_0(\omega, 1-\sigma^{-1}r)))| \leq \frac{C}{\sigma}|\overline{D}w_0(\omega, 1-\sigma^{-1}r)|. \tag{3.9.18}$$

Hence, as a consequence of (3.9.17) and (3.9.18), we have

$$\begin{aligned}
&\sigma^{1-m-\beta}\int_{B_\sigma^+(y)} x_{m+1}^\beta |\nabla w|^2 \mathrm{d}x \\
&\leq C\sigma^{-(1+m+\beta)}\int_{B_\sigma^+(y)\setminus B_{(1-\varepsilon)\sigma}^+(y)} x_{m+1}^\beta |\overline{D}w_0(\omega, 1-\sigma^{-1}r)|^2 \mathrm{d}x. \qquad (3.9.19)
\end{aligned}$$

We paramatrise $\omega \in \mathbb{S}_+^m$ by spherical coordinates and let $r = |x-y|$. Using the change of variables $x = r\omega + y$ we calculate

$$\begin{aligned}
&\int_{B_\sigma^+(y)\setminus B_{(1-\varepsilon)\sigma}^+(y)} x_{m+1}^\beta |\overline{D}w_0(\omega, 1-\sigma^{-1}r)|^2 \mathrm{d}x \\
&= \int_{(1-\varepsilon)\sigma}^{\sigma} r^{m+\beta}\int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |\overline{D}w_0(\omega, 1-\sigma^{-1}r)|^2 \mathrm{d}\omega \mathrm{d}r \\
&\leq \sigma^{m+\beta}\int_{(1-\varepsilon)\sigma}^{\sigma}\int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |\overline{D}w_0(\omega, 1-\sigma^{-1}r)|^2 \mathrm{d}\omega \mathrm{d}r. \qquad (3.9.20)
\end{aligned}$$

The change of variables $s = 1 - \frac{r}{\sigma}$ yields

$$\int_{(1-\varepsilon)\sigma}^{\sigma}\int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |\overline{D}w_0(\omega, 1-\sigma^{-1}r)|^2 \mathrm{d}\omega \mathrm{d}r = \sigma\int_{\mathbb{S}_+^m\times[0,\varepsilon]} \omega_{m+1}^\beta |\overline{D}w_0(\omega, s)|^2 \mathrm{d}\omega \mathrm{d}s. \tag{3.9.21}$$

Thus, we combine (3.9.19), (3.9.20) and (3.9.21) with (3.9.12) to see that

$$\sigma^{1-m-\beta}\int_{B_\sigma^+(y)} x_{m+1}^\beta |\nabla w|^2 \mathrm{d}x \leq C\varepsilon\int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |D\hat{v}|^2 \mathrm{d}\omega + \frac{C}{\varepsilon}\int_{\mathbb{S}_+^m} \omega_{m+1}^\beta |\hat{v}-\lambda|^2 \mathrm{d}\omega.$$

This inequality, together with (3.9.10) and the combination of (3.9.9), (3.9.11) and (3.9.14), implies (3.9.3) as required. □

Remark 3.9.2.1. We note that the function w constructed in Lemma 3.9.2.1 is defined in such a way as to permit the direct comparison of its energy with that of a map v, which minimises E^β relative to $\mathcal{O}$. In particular, we may replace the energy of w with the energy of such a v on the left hand side of (3.9.3). Such a w is often called a comparison function.

3.10 Improved Control in the Poincaré Inequality

The purpose of constructing the comparison function w in Section 3.9 is so we may compare the energy of w to the energy of a v which is a minimising harmonic map relative to $\mathcal{O}$. The end goal is to prove a decay estimate of the form (3.7.3) assuming that the energy of v is small. In order to achieve this we must show that under the same smallness assumption on the energy of v, we also have sufficient control of $\frac{C}{\varepsilon}\rho^{-(1+m+\beta)}\int_{B^+_\rho(y)} x^\beta_{m+1}|v-\overline{v}_{B^+_\rho(y),\beta}|^2\mathrm{d}x$ in (3.9.3). In Section 3.7 we stated that an estimate of the form (3.7.2) would suffice and we prove such an estimate here.

Lemma 3.10.0.1. *For every $\delta > 0$ and every $c_0 > 0$ there exist two constants $\varepsilon = \varepsilon(m, n, \delta, c_0) > 0$ and $\theta = \theta(m, n, \delta, c_0) \in (0, \frac{1}{4}]$ such that the following holds. Let $B^+_R(x_0) \subset \mathbb{R}^{m+1}_+$ be a half-ball with $R \leq 1$ and $(x_0)_{m+1} = 0$. Suppose $v \in W^{1,2}_\beta(B^+_R(x_0);\mathbb{R}^n)$ satisfies*

$$\left|\int_{B^+_R(x_0)} x^\beta_{m+1}\left\langle \nabla v, \nabla\phi\right\rangle \mathrm{d}x\right| \leq c_0\int_{B^+_R(x_0)} x^\beta_{m+1}\left|\phi\right|\left|\nabla v\right|^2\mathrm{d}x$$

for every $\phi \in C^\infty_0(B_R(x_0);\mathbb{R}^n)$. If

$$R^{1-m-\beta}\int_{B^+_R(x_0)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x \leq \varepsilon$$

then

$$(\theta R)^{-(1+m+\beta)}\int_{B^+_{\theta R}(x_0)} x^\beta_{m+1}\left|v-\overline{v}_{B^+_{\theta R}(x_0),\beta}\right|^2\mathrm{d}x \leq \delta R^{1-m-\beta}\int_{B^+_R(x_0)} x^\beta_{m+1}\left|\nabla v\right|^2\mathrm{d}x.$$

Proof. First we observe that the statement of the lemma is invariant under rescaling and translation by any point in $\partial\mathbb{R}^{m+1}_+$; if the lemma is true on $B^+_1(0)$ then we may obtain the lemma on any $B^+_R(x_0) \subset \mathbb{R}^{m+1}_+$ with $R \leq 1$ and $(x_0)_{m+1} = 0$ by rescaling using the map $x \mapsto Rx + x_0$, defined for $x \in B^+_1(0)$, and applying the lemma on $B^+_1(0)$. Thus we will assume $R = 1$ and $x_0 = 0$.

We argue by contradiction. In particular, we will show that if the lemma were false, then we may construct a weak solution of 2.4.5, that is a weak solution of $\mathrm{div}(x^\beta_{m+1}\nabla w) = 0$ in $B^+_1(0)$ with $x^\beta_{m+1}\frac{\partial w}{\partial x_{m+1}} = 0$ on $\partial^0 B^+_1(0)$, whose $L^2_\beta(B^+_1(0);\mathbb{R}^n)$ norm is bounded below and strictly above by the same number, a contradiction.

Now suppose, for a contradiction, that there exist $\delta > 0$ and $c_0 > 0$ such that the claim is false. Then for any $\theta \in (0, \frac{1}{4}]$ there is a sequence of maps $(v_k)_{k\in\mathbb{N}}$, with $v_k \in W^{1,2}_\beta(B^+_1(0);\mathbb{R}^n)$ for every k, such that

$$\left| \int_{B^+_1(0)} x^\beta_{m+1} \langle \nabla v_k, \nabla\phi\rangle \,\mathrm{d}x \right| \leq c_0 \int_{B^+_1(0)} x^\beta_{m+1} |\phi||\nabla v_k|^2 \mathrm{d}x \tag{3.10.1}$$

for every $\phi \in C^\infty_0(B_1(0);\mathbb{R}^n)$ and

$$\int_{B^+_1(0)} x^\beta_{m+1} |\nabla v_k|^2 \mathrm{d}x := \varepsilon_k \to 0 \text{ as } k \to \infty$$

but

$$\begin{aligned} \theta^{-(1+m+\beta)} \int_{B^+_\theta(0)} x^\beta_{m+1} |v_k - \overline{(v_k)}_{B^+_\theta(0),\beta}|^2 \mathrm{d}x &> \delta \int_{B^+_1(0)} x^\beta_{m+1} |\nabla v_k|^2 \mathrm{d}x \\ &= \delta\varepsilon_k. \end{aligned} \tag{3.10.2}$$

In order to construct a weak solution of the Neumann type problem (2.4.5) with the desired properties, we define the normalised sequence $(w_k)_{k\in\mathbb{N}}$ by

$$w_k = \varepsilon_k^{-\frac{1}{2}} (v_k - \overline{(v_k)}_{B^+_\theta(0),\beta})$$

and analyse the limit as $k \to \infty$. We calculate

$$\nabla w_k = \varepsilon_k^{-\frac{1}{2}} \nabla v_k.$$

Hence

$$\int_{B^+_1(0)} x^\beta_{m+1} |\nabla w_k|^2 \mathrm{d}x = 1 \tag{3.10.3}$$

and

$$\overline{(w_k)}_{B^+_\theta(0),\beta} = \frac{1}{\int_{B^+_\theta(0)} x^\beta_{m+1} \mathrm{d}x} \int_{B^+_\theta(0)} x^\beta_{m+1} w_k \mathrm{d}x = 0. \tag{3.10.4}$$

Furthermore, we deduce from (3.10.2) that

$$\theta^{-(1+m+\beta)} \int_{B^+_\theta(0)} x^\beta_{m+1} |w_k|^2 \,\mathrm{d}x > \delta. \tag{3.10.5}$$

Using (3.10.4) we write $w_k = w_k - \overline{(w_k)}_{B^+_\theta(0),\beta}$ and an application of lemma

(2.3.3.4) shows that

$$\int_{B_1^+(0)} x_{m+1}^{\beta}|w_k|^2\mathrm{d}x = \int_{B_1^+(0)} x_{m+1}^{\beta}|w_k - \overline{(w_k)}_{B_\theta^+(0),\beta}|^2\mathrm{d}x$$
$$\leq C\theta^{-(1+m+\beta)} \int_{B_1^+(0)} x_{m+1}^{\beta} \left|\nabla w_k\right|^2 \mathrm{d}x. \tag{3.10.6}$$

Together, (3.10.3) and (3.10.6) show that $(w_k)_{k\in\mathbb{N}}$ is bounded $W_\beta^{1,2}(B_1^+(0);\mathbb{R}^n)$. Hence, the Compactness lemma, Lemma 2.3.5.2, yields a subsequence $(w_{k_j})_{j\in\mathbb{N}}$ which converges weakly in $W_\beta^{1,2}(B_1^+(0);\mathbb{R}^n)$ and strongly in $L_\beta^2(B_1^+(0);\mathbb{R}^n)$ to some $w \in W_\beta^{1,2}(B_1^+(0);\mathbb{R}^n)$.

To see that w is a weak solution of the Neumann-type problem 2.4.5 in $B_1^+(0)$ we use (3.10.1). Let $\phi \in C_0^\infty(B_1(0);\mathbb{R}^n)$. In view of (3.10.3) we find

$$\left|\int_{B_1^+(0)} x_{m+1}^{\beta} \langle \nabla w_k, \nabla\phi\rangle \,\mathrm{d}x\right| = \varepsilon_k^{-\frac{1}{2}} \left|\int_{B_1^+(0)} x_{m+1}^{\beta} \langle \nabla v_k, \nabla\phi\rangle \,\mathrm{d}x\right|$$
$$\leq c_0||\phi||_{L^\infty(B_1^+(0);\mathbb{R}^n)}\varepsilon_k^{-\frac{1}{2}} \int_{B_1^+(0)} x_{m+1}^{\beta} \left|\nabla v_k\right|^2 \mathrm{d}x$$
$$= c_0||\phi||_{L^\infty(B_1^+(0);\mathbb{R}^n)}\varepsilon_k^{-\frac{1}{2}}\varepsilon_k \int_{B_1^+(0)} x_{m+1}^{\beta} \left|\nabla w_k\right|^2 \mathrm{d}x$$
$$= c_0||\phi||_{L^\infty(B_1^+(0);\mathbb{R}^n)}\varepsilon_k^{\frac{1}{2}}.$$

Since $w_{k_j} \rightharpoonup w$ in $W_\beta^{1,2}(B_1^+(0);\mathbb{R}^n)$, it follows that

$$\left|\int_{B_1^+(0)} x_{m+1}^{\beta}\langle\nabla w, \nabla\phi\rangle\mathrm{d}x\right| = \lim_{j\to\infty}\left|\int_{B_1^+(0)} x_{m+1}^{\beta}\langle\nabla w_{k_j}, \nabla\phi\rangle\mathrm{d}x\right|$$
$$\leq c_0||\phi||_{L^\infty(B_1^+(0);\mathbb{R}^n)} \lim_{j\to\infty} \varepsilon_{k_j}^{\frac{1}{2}} = 0$$

for every $\phi \in C_0^\infty(B_1(0);\mathbb{R}^n)$. Hence w is a weak solution of (2.4.5) in $B_1^+(0)$.

We also conclude, using the Compactness Lemma, Lemma 2.3.5.2, to take limits in (3.10.3), (3.10.4) and (3.10.5), that

$$\int_{B_1^+(0)} x_{m+1}^{\beta} \left|\nabla w\right|^2 \mathrm{d}x \leq 1, \tag{3.10.7}$$

$$\overline{w}_{B_\theta^+(0),\beta} = \frac{1}{\int_{B_\theta^+(0)} x_{m+1}^{\beta}\mathrm{d}x} \int_{B_\theta^+(0)} x_{m+1}^{\beta} w\mathrm{d}x = 0 \tag{3.10.8}$$

and

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^{\beta} \left|w\right|^2 \mathrm{d}x \geq \delta \tag{3.10.9}$$

respectively. Now, in view of (3.10.8), the Poincaré inequality, Lemma 2.3.3.3, yields

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |w|^2 \mathrm{d}x \leq C\theta^{1-m-\beta} \int_{B_\theta^+(0)} x_{m+1}^\beta |\nabla w|^2 \mathrm{d}x. \tag{3.10.10}$$

Lastly, since w is a weak solution of 2.4.5 we may apply Corollary 2.4.3.2 to w with $\theta \leq \frac{1}{4}$ (so that $2\theta \leq \frac{1}{2}$). This gives a positive constant C (independent of θ) and a $\gamma \in (0,1)$ such that

$$\theta^{1-m-\beta} \int_{B_\theta^+(0)} x_{m+1}^\beta |\nabla w|^2 \mathrm{d}x \leq C(2\theta)^{2\gamma}. \tag{3.10.11}$$

Combining (3.10.10) and (3.10.11) we see that

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |w|^2 \mathrm{d}x \leq C(2\theta)^{2\gamma}. \tag{3.10.12}$$

This holds for all fixed $\theta \in (0, \frac{1}{4}]$ and we choose $\theta < \frac{1}{2}\left(\frac{\delta}{C}\right)^{\frac{1}{2\gamma}}$ so that (3.10.12) contradicts (3.10.9). Hence the lemma is proved. □

3.11 Energy Decay Lemma

If the energy of a minimising harmonic map relative to $\mathcal{O}$ is sufficiently small on a half-ball $B_R^+(x_0)$ with $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$ then we can show that the decay of the energy on smaller, concentric half-balls is slightly better than the decay we obtain from the monotonicity formula, Lemma 3.3.1.1. The subsequent lemma encapsulates this fact and is the final precursor to an ε-regularity result for minimisers of E^β relative to $\mathcal{O}$. The lemma corresponds to (3.7.3) in Section 3.7.

Lemma 3.11.0.1. *Let $\beta \in (-1,1)$ and $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ be a minimiser of E^β relative to $\mathcal{O}$. Suppose $B_R^+(x_0)$ is a half-ball with $R \leq 1$ and $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$. There exist $\varepsilon_0 = \varepsilon_0(m, N, \beta) > 0$ and $\theta_0 = \theta_0(m, N, \beta) \in (0, \frac{1}{4})$ such that if*

$$R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \leq \varepsilon_0,$$

then

$$(\theta_0 r)^{1-m-\beta} \int_{B_{\theta_0 r}^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \leq \frac{1}{2} r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x,$$

for every $B_r^+(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$.

Proof. To prove the lemma we combine Corollary 3.9.2.1 and Lemma 3.10.0.1 in an appropriate way. Our strategy is as follows. We show that if the energy on $B_R^+(x_0)$ is small then it remains proportionally small on any half-ball $B_r^+(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$. Therefore, using the monotonicity formula, Lemma 3.3.1.1, we deduce that if we can apply Corollary 3.9.2.1 on $B_r^+(y)$ then we can apply it on $B_\rho^+(y)$ for $\rho \le r$. We do so for $\rho \le r$ arbitrary and then choose ε small in the corollary. With this choice in mind, we then choose a δ and a c_0 for which to apply Lemma 3.10.0.1 and finally conclude the proof by using our findings to choose ρ accordingly in terms of r. Throughout the proof we will assume that, unless stated otherwise, all constants depend only on m, N and β and only distinguish them when necessary.

Let $B_\rho^+(y) \subset B_r^+(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$. Then $\rho \le r \le \frac{R}{2}$, $y \in \partial\mathbb{R}_+^{m+1}$ and $|x_0 - y| < \frac{R}{2}$. Furthermore, as $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$ we have $y \in \mathcal{O}$. First we note how the energy scales with the radius ρ. Suppose $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ minimises E^β relative to $\mathcal{O}$ and satisfies $R^{1-m-\beta} \int_{B_R^+(x_0)} |\nabla v|^2 \mathrm{d}x \le \varepsilon_0$ for $\varepsilon_0 > 0$ to be chosen as required. Then for any $\rho \in (0, r]$ the monotonicity formula, Lemma 3.3.1.1, yields

$$\begin{aligned}
\rho^{1-m-\beta} \int_{B_\rho^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x &\le r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \\
&\le \left(\frac{R}{2}\right)^{1-m-\beta} \int_{B_{\frac{R}{2}}^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \\
&\le C R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \\
&\le C\varepsilon_0. \qquad (3.11.1)
\end{aligned}$$

We apply Corollary 3.9.2.1 on $B_\rho^+(y) \subset B_r^+(y)$, with $\rho \le r$ to be chosen later. This gives a δ_0 such that for any $\varepsilon \in (0,1)$, if

$$\rho^{1-m-\beta} \int_{B_\rho^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \le \delta_0^2 \varepsilon^{1+m+\frac{\beta}{2}+\frac{|\beta|}{2}} \qquad (3.11.2)$$

then there is a $\sigma \in (\frac{3\rho}{4}, \rho)$ such that we can find a $w_\varepsilon \in W_\beta^{1,2}(B_\rho^+(y); N)$ which agrees with v in $B^+_\rho(y)\backslash B^+_\sigma(y)$ and such that

$$\begin{aligned}&\sigma^{1-m-\beta}\int_{B^+_\sigma(y)} x^\beta_{m+1}|\nabla w|^2 \mathrm{d}x\\ \le\; & C\varepsilon\rho^{1-m-\beta}\int_{B^+_\rho(y)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x + \frac{1}{\varepsilon}C\rho^{-(1+m+\beta)}\int_{B^+_\rho(y)} x^\beta_{m+1}|v-\overline{v}_{B^+_\rho(y),\beta}|^2\mathrm{d}x.\end{aligned} \tag{3.11.3}$$

Assuming (3.11.2) and consequently (3.11.3) hold, we make use of the comparison property of w. Since $v = w$ in $B^+_\rho(y)\backslash B^+_\sigma(y)$ we may extend w to an element of $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ by requiring $w = v$ on $\mathbb{R}^{m+1}_+\backslash B^+_\rho(y)$. Therefore, as v is a minimiser of E^β relative to $\mathcal{O}$, we deduce that

$$\int_{B^+_\sigma(y)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x \le \int_{B^+_\sigma(y)} x^\beta_{m+1}|\nabla w|^2\mathrm{d}x.$$

Combining this fact with the monotonicity formula, Lemma 3.3.1.1, and (3.11.3) gives

$$\begin{aligned}&\left(\frac{3\rho}{4}\right)^{1-m-\beta}\int_{B^+_{\frac{3\rho}{4}}(y)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x\\ \le\; & C\varepsilon\rho^{1-m-\beta}\int_{B^+_\rho(y)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x + \frac{1}{\varepsilon}C\rho^{-(1+m+\beta)}\int_{B^+_\rho(y)} x^\beta_{m+1}|v-\overline{v}_{B^+_\rho(y),\beta}|^2\mathrm{d}x.\end{aligned} \tag{3.11.4}$$

Fix $\varepsilon = \min\{\frac{1}{4}, \frac{1}{4C}\}$, where C is the constant in (3.11.4). We assume henceforth that $\varepsilon_0 \le \frac{1}{C}\delta_0^2\varepsilon^{1+m+\frac{\beta}{2}+\frac{|\beta|}{2}}$ where C is the constant from (3.11.1). It follows from (3.11.1) that (3.11.2) is satisfied and hence, substituting this ε into (3.11.4), we have

$$\begin{aligned}&\left(\frac{3\rho}{4}\right)^{1-m-\beta}\int_{B^+_{\frac{3\rho}{4}}(y)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x\\ \le\; & \frac{1}{4}\rho^{1-m-\beta}\int_{B^+_\rho(y)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x + \hat{C}\rho^{-(1+m+\beta)}\int_{B^+_\rho(y)} x^\beta_{m+1}|v-\overline{v}_{B^+_\rho(y),\beta}|^2\mathrm{d}x\end{aligned} \tag{3.11.5}$$

for a constant $\hat{C}$ and any $\rho \le r \le \frac{R}{2}$. Now we apply Lemma 3.10.0.1 with

$\delta = \min\{\frac{1}{4}, \frac{1}{4\hat{C}}\}$ fixed, depending on m, N and β. Observe that

$$\left|\int_{B_r^+(y)} x_{m+1}^\beta \langle \nabla v, \nabla \phi \rangle \, \mathrm{d}x\right| = \left|\int_{B_r^+(y)} x_{m+1}^\beta \langle \phi, A(v)(\nabla v, \nabla v) \rangle \, \mathrm{d}x\right|$$
$$\leq c_0 \int_{B_r^+(y)} x_{m+1}^\beta |\phi||\nabla v|^2 \mathrm{d}x, \tag{3.11.6}$$

for every $\phi \in C_0^\infty(B_r(y); \mathbb{R}^n)$ on every $B_r^+(y) \subset B_R^+(x_0)$ where $c_0 = c_0(m, N)$. Hence, we may apply the lemma for δ, c_0 as above to obtain a corresponding $\varepsilon_1 > 0$ and $\theta_1 \in (0, \frac{1}{4}]$ such that if $r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \leq \varepsilon_1$ then

$$(\theta_1 r)^{-(1+m+\beta)} \int_{B_{\theta_1 r}^+(y)} x_{m+1}^\beta |v - \overline{v}_{B_{\theta_1 r}^+(y),\beta}|^2 \mathrm{d}x \leq \frac{1}{4\hat{C}} r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x. \tag{3.11.7}$$

Now choose $\varepsilon_0 = \frac{1}{C} \min\{\delta_0^2 \varepsilon^{1+m+\frac{\beta}{2}+\frac{|\beta|}{2}}, \varepsilon_1\}$ where C is the constant from (3.11.1). It follows that (3.11.5) and (3.11.7) hold on any $B_\rho^+(y) \subset B_r^+(y) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$. Thus we may apply (3.11.5) with $\rho = \theta_1 r$. In turn, assuming this choice of ρ, we combine (3.11.5) with the monotonicity formula and (3.11.7) to see that

$$\left(\frac{3\theta_1 r}{4}\right)^{1-m-\beta} \int_{B_{\frac{3\theta_1 r}{4}}^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \leq \frac{1}{2} r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x.$$

Hence the lemma is proved with the above choice of ε_0 and $\theta_0 = \frac{3\theta_1}{4}$. □

3.12 Hölder Continuity of Energy Minimisers

Here we present an ε-regularity theorem for minimisers of E^β relative to $\mathcal{O}$ which is a culmination of all our preceding results. That is, we show that if the energy of a minimiser is sufficiently small in a given half-ball then the minimiser is Hölder continuous in a smaller concentric half-ball. The following lemma is an analogue of the regularity estimate, Theorem 3.1 in [44], due to Schoen and Uhlenbeck for harmonic maps between Riemannian manifolds. Since we have not prescribed boundary data, the theorem has the form of an interior regularity result.

3.12.1 ε-regularity Theorem

Theorem 3.12.1.1. *Let $\beta \in (-1, 1)$ and $v \in \dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1}; N)$ be a minimiser of E^β relative to $\mathcal{O}$. Suppose $B_R^+(x_0)$ is a half-ball with $R \leq 1$ and $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$.*

There exists an $\varepsilon = \varepsilon(m, N, \beta) > 0$ and a $\theta = \theta(m, N, \beta) \in (0, 1)$ such that if

$$R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \le \varepsilon,$$

then $v \in C^{0,\gamma}(\overline{B_{\theta R}^+(x_0)}; N)$ for some $\gamma = \gamma(m, N, \beta) \in (0, 1)$. In particular,

$$|v(x_1) - v(x_2)| \le C \left(R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\frac{|x_1 - x_2|}{R} \right)^\gamma \tag{3.12.1}$$

for every $x_1, x_2 \in B_{\theta R}^+(x_0)$ and a constant $C = C(m, N, \beta)$.

Proof. To prove the theorem, we show that minimisers of E^β relative to $\mathcal{O}$ satisfy (3.5.3) and (3.5.4) and apply Lemma 3.5.0.2. To achieve this we choose ε small enough so we may apply Lemma 3.6.0.4 and Lemma 3.11.0.1. We conclude minimisers satisfy (3.5.3) by applying Lemma 3.11.0.1 and Lemma 3.5.1.1. This fact, together with an application of Lemma 3.6.0.4 shows minimisers satisfy (3.5.4). Throughout the proof we adopt the convention that all constants depend only on m, N and β unless stated otherwise. We reinforce this dependence where appropriate.

Let v be a minimiser of E^β relative to $\mathcal{O}$ with $R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \le \varepsilon$ for $\varepsilon = \min\{\varepsilon_0, \varepsilon_1\}$, where ε_0 is the number from Lemma 3.11.0.1 and ε_1 is the number from lemma 3.6.0.4. We now proceed to investigate the implications of this choice of ε in terms of the application of the aforementioned lemmata.

The choice of ε allows us to apply Lemma 3.11.0.1. This yields a $\theta_0 \in (0, \frac{1}{4}]$ such that on every $B_{\tilde{r}}^+(z) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$ we have

$$(\theta_0 \tilde{r})^{1-m-\beta} \int_{B_{\theta_0 \tilde{r}}^+(z)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \le \frac{1}{2} \tilde{r}^{1-m-\beta} \int_{B_{\tilde{r}}^+(z)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x. \tag{3.12.2}$$

Hence, since the function $f(\tilde{r}) = \tilde{r}^{1-m-\beta} \int_{B_{\tilde{r}}^+(z)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x$ is non-decreasing on $(0, \frac{R}{2}]$ by the monotonicity formula, Lemma 3.3.1.1, we may apply Lemma 3.5.1.1 (with $h = 0$) to this f. We deduce that for any $B_{\tilde{r}}^+(z) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$ we have

$$\begin{aligned} \tilde{r}^{1-m-\beta} \int_{B_{\tilde{r}}^+(z)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x &\le C \left(2 \frac{\tilde{r}}{R} \right)^{\gamma_0} \left(\frac{R}{2} \right)^{1-m-\beta} \int_{B_{\frac{R}{2}}^+(z)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \\ &\le C \left(\frac{\tilde{r}}{R} \right)^{\gamma_0} R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \end{aligned} \tag{3.12.3}$$

for a constant C and a $\gamma_0 \in (0, 1)$ which depend on m, N, β and θ_0, and hence only on m, N, β. Thus, v satisfies (3.5.3) in Lemma 3.5.0.2 for all $B^+_{\tilde{r}}(z) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$.

We now focus on a proof of (3.5.4). It follows from our choice of ε that we may apply Lemma 3.6.0.4. Hence, for any $B_r(y) \in \mathcal{B}_{\theta_1}(x_0, R, \frac{R}{3})$, with $\theta_1 \geq 2$ given by the lemma, and any $0 < \rho \leq r$ we have

$$\rho^{1-m} \int_{B_\rho(y)} |\nabla v|^2 \, \mathrm{d}x \leq C \left(\frac{\rho}{r}\right)^{\gamma_1} r^{1-m} \int_{B_r(y)} |\nabla v|^2 \, \mathrm{d}x \tag{3.12.4}$$

for some $\gamma_1 \in (0,1)$. In order to verify that (3.5.4) is satisfied we combine (3.12.4) for $r = \frac{y_{m+1}}{\theta_1}$, with (3.12.3) applied on a suitable choice of half-ball $B^+_{\tilde{r}}(z)$. Since $\theta_1 \geq 2$, for any $B_r(y) \in \mathcal{B}_{\theta_1}(x_0, R, \frac{R}{3})$ we have the inclusions

$$B_r(y) \subset B_{\frac{y_{m+1}}{\theta_1}}(y) \subset B^+_{\left(\frac{\theta_1+1}{\theta_1}\right) y_{m+1}}(y^+) \subset B^+_{\frac{3y_{m+1}}{2}}(y^+) \in \mathcal{B}^+\left(x_0, R, \frac{R}{2}\right). \tag{3.12.5}$$

These inclusions, together with (3.4.3) from Section 3.4, yield

$$\begin{aligned}
&\left(\frac{y_{m+1}}{\theta_1}\right)^{1-m} \int_{B_{\frac{y_{m+1}}{\theta_1}}(y)} |\nabla v|^2 \, \mathrm{d}x \\
&\leq C \left(\frac{y_{m+1}}{\theta_1}\right)^{1-m-\beta} \int_{B^+_{\left(\frac{\theta_1+1}{\theta_1}\right) y_{m+1}}(y^+)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \\
&\leq C \left(\frac{(\theta_1+1) y_{m+1}}{\theta_1}\right)^{1-m-\beta} \int_{B^+_{\left(\frac{\theta_1+1}{\theta_1}\right) y_{m+1}}(y^+)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x,
\end{aligned} \tag{3.12.6}$$

where C depends on m, N, β and θ_1 and thus only on m, N, β. In view of (3.12.5) and (3.12.6) we note that $B^+_{\tilde{r}}(z)$, with $\tilde{r} = \left(\frac{\theta_1+1}{\theta_1}\right) y_{m+1}$ and $z = y^+$, is a sufficient choice of half-ball on which to apply (3.12.3). Let $\hat{\gamma} = \min\{\gamma_0, \gamma_1\}$. We combine (3.12.3), applied on $B^+_{\left(\frac{\theta_1+1}{\theta_1}\right) y_{m+1}}(y^+)$, with (3.12.4), applied on $B_{\frac{y_{m+1}}{\theta_1}}(y)$, and

(3.12.6) to see that

$$
\begin{aligned}
&\rho^{1-m}\int_{B_\rho(y)}|\nabla v|^2\,\mathrm{d}x\\
&\leq C\left(\frac{\theta_1\rho}{y_{m+1}}\right)^{\hat\gamma}\left(\frac{y_{m+1}}{\theta_1}\right)^{1-m}\int_{B_{\frac{y_{m+1}}{\theta_1}}(y)}|\nabla v|^2\,\mathrm{d}x\\
&\leq C\left(\frac{\rho}{y_{m+1}}\right)^{\hat\gamma}\left(\frac{(\theta_1+1)y_{m+1}}{\theta_1}\right)^{1-m-\beta}\int_{B^+_{\left(\frac{\theta_1+1}{\theta_1}\right)y_{m+1}}(y^+)}x_{m+1}^\beta|\nabla v|^2\mathrm{d}x\\
&\leq C\left(\frac{\rho}{y_{m+1}}\right)^{\hat\gamma}\left(\frac{y_{m+1}}{R}\right)^{\hat\gamma}R^{1-m-\beta}\int_{B^+_R(x_0)}x_{m+1}^\beta|\nabla v|^2\mathrm{d}x\\
&\leq C\left(\frac{\rho}{R}\right)^{\hat\gamma}R^{1-m-\beta}\int_{B^+_R(x_0)}x_{m+1}^\beta|\nabla v|^2\mathrm{d}x.
\end{aligned}
\tag{3.12.7}
$$

This holds for any $B_\rho(y)\in\mathcal{B}_{\theta_1}(x_0,R,\frac{R}{3})$ and since (3.12.3) holds on $\mathcal{B}^+(x_0,R,\frac{R}{2})$, it holds on $\mathcal{B}^+(x_0,R,\frac{R}{3})$. Thus we have satisfied the assumptions (3.5.3) and (3.5.4) of Lemma 3.5.0.2. Applying this lemma concludes the proof. □

3.12.2 Partial Regularity for Energy Minimisers

We now provide an estimate for the Hausdorff dimension, with respect to the Euclidean metric, of the set of points in $\mathcal{O}$ where a minimiser of E^β relative to $\mathcal{O}$ fails to be Hölder continuous. Combined with Theorem 3.12.1.1 this yields a partial regularity result for minimisers of E^β relative to $\mathcal{O}$ which constitutes Theorem 3.2.0.1 in Section 3.2.

Lemma 3.12.2.1. *Suppose $v\in\dot W^{1,2}_\beta(\mathbb{R}^{m+1}_+;N)$ is a minimiser of E^β relative to $\mathcal{O}$. Let $\Sigma_{int}\subset\mathbb{R}^{m+1}_+$ denote the set of points, discussed in Section 3.2, which is relatively closed in $\mathbb{R}^{m+1}_+$ and has Hausdorff dimension $m-2$, such that v is smooth in $\mathbb{R}^{m+1}_+\backslash\Sigma_{int}$. Then there exist $\gamma\in(0,1)$ and a relatively closed set $\Sigma_{bdry}\subset\mathcal{O}$ of vanishing $m+\beta-1$-dimensional Hausdorff measure, with respect to the Euclidean metric, such that $v\in C^{0,\gamma}((\mathbb{R}^{m+1}_+\cup\mathcal{O})\backslash\Sigma;N)$ where $\Sigma=\Sigma_{int}\cup\Sigma_{bdry}$. Furthermore Σ is relatively closed in $\mathbb{R}^{m+1}_+\cup\mathcal{O}$ and $\mathcal{H}^{m+\beta-1}(\Sigma)=0$.*

Proof. The proof closely follows the proof of Theorem 3.2 in [33]; Theorem 3.12.1.1 indicates a choice for Σ_{bdry} and we use a covering argument to estimate the Hausdorff dimension of this set.

The ε-regularity result, Theorem 3.12.1.1, gives us a candidate for Σ_{bdry}; define

$$\Sigma_{\mathrm{bdry}}=\{y\in\mathcal{O}:\Theta_v^\beta(y)\geq\varepsilon\}$$

where ε is the number given by the theorem and

$$\Theta_v^\beta(y) = \lim_{r\to 0^+} r^{1-m-\beta}\int_{B_r^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x$$

is the density function defined in Section 3.3.1. It follows from Lemma 3.12.1.1 that every point in $\mathcal{O}\backslash\Sigma_{\text{bdry}}$ is contained in a neighbourhood where v is Hölder continuous. Moreover, in view of Lemma 3.3.1.2, Θ_v^β is upper semi-continuous which, when combined with the definition of Σ_{bdry}, shows that Σ_{bdry} is relatively closed in $\mathcal{O}$.

We want to show that the $m+\beta-1$ dimensional Hausdorff measure of Σ_{bdry} vanishes. We may write Σ_{bdry} as a countable union of compact sets of the form $K\cap\Sigma_{\text{bdry}}$ where $K\subset\mathcal{O}$ is compact. Let $\Sigma'\subset\Sigma_{\text{bdry}}$ be such a set. Fix $\delta>0$ and cover Σ' by a collection of balls $B_{r_i}^m(x_i)\subset\mathcal{O}$ with $\overline{B_{r_i}^m(x_i)}\subset\mathcal{O}$ with $x_i\in\Sigma'$ and $0<r_i\le\delta$. The compactness of Σ', combined with Vitali's covering theorem yields a finite subcollection of balls, $B_{r_1}^m(x_1),\ldots,B_{r_I}^m(x_I)$ for some $I\in\mathbb{N}$, of any such cover of Σ', which satisfies

$$B_{r_i}(x_i)\cap B_{r_j}(x_j)=\emptyset \quad \text{for } i\neq j, 1\le i,j\le I$$

and

$$\Sigma'\subset\bigcup_{i=1}^I B_{5r_i}(x_i).$$

For each i, the boundary energy monotonicitly formula, Lemma 3.3.1.1, implies

$$\varepsilon\le r_i^{1-m-\beta}\int_{B_{r_i}^+(x_i)} x_{m+1}^\beta|\nabla v|^2\mathrm{d}x.$$

Hence,

$$\begin{aligned}\sum_{i=1}^I (10r_i)^{m+\beta-1} &\le \frac{10^{m+\beta-1}}{\varepsilon}\sum_{i=1}^I\int_{B_{r_i}^+(x_i)} x_{m+1}^\beta|\nabla v|^2\mathrm{d}x\\ &\le \frac{10^{m+\beta-1}}{\varepsilon}\int_{\mathcal{O}\times[0,\delta]} x_{m+1}^\beta|\nabla v|^2\mathrm{d}x. \qquad (3.12.8)\end{aligned}$$

Using Lebesgue's Dominated Convergence Theorem we send $\delta\to 0^+$ in (3.12.8). This shows that $\mathcal{H}^{m-1+\beta}(\Sigma')=0$ and hence $\mathcal{H}^{m-1+\beta}(\Sigma_{\text{bdry}})=0$.

Now we show that $\Sigma=\Sigma_{\text{int}}\cup\Sigma_{\text{bdry}}$ is relatively closed in $\mathbb{R}_+^{m+1}\cup\mathcal{O}$. Let $x_0\in(\mathbb{R}_+^{m+1}\cup\mathcal{O})\backslash\Sigma$. If $x_0\in\mathbb{R}_+^{m+1}$ then $x_0\in\mathbb{R}_+^{m+1}\backslash\Sigma_{\text{int}}$ and the discussion in Section 3.2 after Theorem 3.2.0.1 implies that v is smooth in an open ball centred at x_0 and contained in $\mathbb{R}^{m+1}_+\backslash\Sigma_{\mathrm{int}} \subset (\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$. If $x_0 \in \mathcal{O}$ then $x_0 \in \mathcal{O}\backslash\Sigma_{\mathrm{bdry}}$ and $\Theta^\beta_v(x_0) < \varepsilon$ which, combined with the fact that $\mathcal{O}\backslash\Sigma_{\mathrm{bdry}}$ is open in $\mathcal{O}$, implies there exists an $R > 0$ such that $R^{1-m-\beta}\int_{B^+_R(x_0)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x \leq \varepsilon$, $R \leq 1$ and $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}\backslash\Sigma_{\mathrm{bdry}}$. Consequently, Theorem 3.12.1.1 implies that there are $\theta, \gamma \in (0, 1)$ such that $v \in C^{0,\gamma}(\overline{B^+_{\theta R}(x_0)}; N)$. Furthermore, we deduce from (3.12.3) in the proof of Theorem 3.12.1.1 that

$$r^{1-m-\beta}\int_{B^+_r(z)} x^\beta_{m+1}|\nabla v|^2\mathrm{d}x \leq C\left(\frac{r}{R}\right)^\gamma \varepsilon$$

on every $B^+_r(z) \in \mathcal{B}^+(x_0, R, \frac{R}{2})$ which shows that $\Theta^\beta_v(z) = 0$ for every $z \in \partial^0 B^+_{\frac{R}{2}}(x_0)$. Now setting $\sigma = \min\{\theta, \frac{1}{2}\}$ we see that $\Theta^\beta_v(z) = 0$ for $z \in \partial^0 B^+_{\sigma R}(x_0)$ which implies $\partial^0 B^+_{\sigma R}(x_0) \subset \mathcal{O}\backslash\Sigma_{\mathrm{bdry}}$. Furthermore, v is a Hölder continuous weakly harmonic map in any $B_r(y)$ with $\overline{B_r(y)} \subset B^+_{\sigma R}(x_0)$. We apply Lemma 3.6.0.2 to see that v is smooth in $B^+_{\sigma R}(x_0)$ and conclude that $B^+_{\sigma R}(x_0) \subset \mathbb{R}^{m+1}_+\backslash\Sigma_{\mathrm{int}}$. Consequently, we have $B^+_{\sigma R}(x_0) \cup \partial^0 B^+_{\sigma R}(x_0) \subset (\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$ and note that $B^+_{\sigma R}(x_0) \cup \partial^0 B^+_{\sigma R}(x_0)$ is an open ball centred at x_0 in the (Euclidean) topology of $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$. Hence Σ is relatively closed in $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$. We also conclude $v \in C^{0,\gamma}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma; N)$.

Finally, as the Hausdorff dimension of Σ_{int} is $m - 2 < m - 1 + \beta$ and $\mathcal{H}^{m-1+\beta}(\Sigma_{\mathrm{bdry}}) = 0$, we deduce that $\mathcal{H}^{m+\beta-1}(\Sigma) = 0$. □

Remark 3.12.2.1. We define $\Sigma_{\mathrm{bdry}} := \{y \in \mathcal{O} : \Theta^\beta_v(y) \geq \varepsilon\}$ henceforth.

Chapter 4

Preliminaries for Regularity of First Order Derivatives

Monotonicity formulas and related properties of solutions of second order linear PDEs are a vital component in the regularity theory for semi-linear second order PDEs, where the highest order term is linear. The main topic of discussion here will be the validity of certain monotonicity formula for the average energy of weak solutions, and their derivatives, of $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$, particularly in the context of the Neumann type problem (2.4.5) or the Dirichlet problem as described below in Section 4.2.

The following monotonicity formula is asserted in [6] for balls with centre $(0', x_{m+1}) \in \mathbb{R}^{m+1}$; we state the theorem on $B_1(0) \subset \mathbb{R}^{m+1}$ which is a particular case of their result.

Lemma 4.0.0.1 (Particular Case of Theorem 2.6 of [6])**.** *Suppose v is a weak solution of div$(|x_{m+1}|^\beta \nabla v) = 0$ in $B_1(0)$ with $v \in W^{1,2}_\beta(B_1(0);\mathbb{R}^n)$ and $v \in L^2(B_1^+(0);\mathbb{R}^n)$. Then for every $0 < r \leq R < 1$ it follows that*

$$r^{-(1+m+\beta)} \int_{B_r(0)} |x_{m+1}|^\beta \, |\nabla v|^2 \, \mathrm{d}x \leq R^{-(1+m+\beta)} \int_{B_R(0)} |x_{m+1}|^\beta \, |\nabla v|^2 \, \mathrm{d}x. \quad (4.0.1)$$

We claim that this is actually not true if $\beta \in (0,1)$ unless the solution v is also symmetric with respect to the hyperplane $\mathbb{R}^m \times \{0\}$, that is $v(x', x_{m+1}) = v(x', -x_{m+1})$ for every $x = (x', x_{m+1})$ in $B_1(0)$. We will show that the formula holds for all $\beta \in (-1,1)$ given the aforementioned symmetry and provide a counter example for general v below.

Lemma 4.0.0.2. *The function*

$$\tilde{v}(x', x_{m+1}) = \begin{cases} \frac{1}{1-\beta} x_{m+1}^{1-\beta} & \text{if } x_{m+1} \geq 0 \\ -\frac{1}{1-\beta}(-x_{m+1})^{1-\beta} & \text{if } x_{m+1} < 0 \end{cases}$$

is a weak solution of $\operatorname{div}(|x_{m+1}|^{\beta} \nabla \tilde{v}) = 0$ *in* $B_1(0)$ *and satisfies the assumptions of Lemma 4.0.0.1 for every* $\beta \in (-1, 1)$*. When* $\beta \in (0, 1)$ *the map* $\tilde{v}$ *does not satisfy the conclusion of Lemma 4.0.0.1.*

The function $\tilde{v}$ defined in the preceding lemma is the odd reflection of the function $\frac{1}{1-\beta} x_{m+1}^{1-\beta}$ in the hyperplane $\mathbb{R}^m \times \{0\}$ which, as we will show later, is a solution of $\operatorname{div}(x_{m+1}^{\beta} \nabla v) = 0$ in bounded domains in $\mathbb{R}^{m+1}_+$. To establish that $\operatorname{div}(|x_{m+1}|^{\beta} \nabla \tilde{v}) = 0$ in $B_1(0)$, we prove the following, more general fact which we will also use in the proof of the monotonicity formula.

Lemma 4.0.0.3. *Let* $R > 0$, $x_0 \in \partial \mathbb{R}^{m+1}_+$ *and suppose* $v \in W^{1,2}_{\beta}(B^+_R(x_0); \mathbb{R}^n)$ *is a weak solution of* (2.4.1) *in* $B^+_R(x_0)$*. Let* $Tv = 0$ *on* $\partial^0 B^+_R(x_0)$ *where* T *is a bounded linear trace operator with the properties described in Section 2.3.2. Then the odd reflection of* v *in* $\mathbb{R}^m \times \{0\}$*, that is, the function*

$$\tilde{v}(x', x_{m+1}) = \begin{cases} v(x', x_{m+1}) & \text{if } x_{m+1} \geq 0 \\ -v(x', -x_{m+1}) & \text{if } x_{m+1} < 0 \end{cases}$$

is in $W^{1,2}_{\beta}(B_R(x_0); \mathbb{R}^n)$ *and is a weak solution of* (2.4.1) *in* $B_R(x_0)$*. Moreover, the weak derivatives of* $\tilde{v}$ *are given by the formula*

$$\frac{\partial \tilde{v}}{\partial x_i}(x', x_{m+1}) = sgn(x_{m+1}) \frac{\partial v}{\partial x_i}(x', |x_{m+1}|) \tag{4.0.2}$$

for $i = 1, \dots, m$ *and*

$$\frac{\partial \tilde{v}}{\partial x_{m+1}}(x', x_{m+1}) = \frac{\partial v}{\partial x_{m+1}}(x', |x_{m+1}|). \tag{4.0.3}$$

Proof. To prove the lemma we need to show that $\tilde{v} \in W^{1,2}_{\beta}(B_R(x_0); \mathbb{R}^n)$ and that $\tilde{v}$ is a weak solution of $\operatorname{div}(|x_{m+1}|^{\beta} \nabla \tilde{v}) = 0$ in $B_R(x_0)$.

First we show that the weak partial derivatives of $\tilde{v}$ are defined by (4.0.2) and (4.0.3). Let $B^-_R(x_0) = \mathbb{R}^{m+1}_- \cap B_R(x_0)$ and $\phi \in C^{\infty}_0(B_R(x_0); \mathbb{R}^n)$. Then for

$i = 1, \ldots, m$ we calculate

$$\int_{B_R(x_0)} \left\langle \operatorname{sgn}(x_{m+1}) \frac{\partial v}{\partial x_i}(x', |x_{m+1}|), \phi \right\rangle \mathrm{d}x = \int_{B_R^+(x_0)} \left\langle \frac{\partial v}{\partial x_i}(x', x_{m+1}), \phi \right\rangle \mathrm{d}x - \int_{B_R^-(x_0)} \left\langle \frac{\partial v}{\partial x_i}(x', -x_{m+1}), \phi \right\rangle \mathrm{d}x. \tag{4.0.4}$$

We consider the integrals over the regions $B_R^+(x_0)$ and $B_R^-(x_0)$ separately. Since ϕ vanishes on $\partial^+ B_R^+(x_0)$, an integration by parts yields

$$\int_{B_R^+(x_0)} \left\langle \frac{\partial v}{\partial x_i}, \phi \right\rangle \mathrm{d}x = -\int_{B_R^+(x_0)} \left\langle \tilde{v}, \frac{\partial \phi}{\partial x_i} \right\rangle \mathrm{d}x. \tag{4.0.5}$$

Similarly, the change of variables $x_{m+1} \mapsto -x_{m+1}$, followed by an integration by parts and the same change of variables again, gives

$$\begin{aligned} -\int_{B_R^-(x_0)} \left\langle \frac{\partial v}{\partial x_i}(x', -x_{m+1}), \phi \right\rangle \mathrm{d}x &= -\int_{B_R^+(x_0)} \left\langle \frac{\partial v}{\partial x_i}, \phi(x', -x_{m+1}) \right\rangle \mathrm{d}x \\ &= \int_{B_R^+(x_0)} \left\langle v, \frac{\partial \phi}{\partial x_i}(x', -x_{m+1}) \right\rangle \mathrm{d}x \\ &= -\int_{B_R^-(x_0)} \left\langle \tilde{v}, \frac{\partial \phi}{\partial x_i} \right\rangle \mathrm{d}x. \end{aligned} \tag{4.0.6}$$

Together, (4.0.4), (4.0.5) and (4.0.6) show that the weak partial derivatives $\frac{\partial \tilde{v}}{\partial x_i}$ are defined by (4.0.2) for $i = 1, \ldots, m$.

Now we show that $\frac{\partial \tilde{v}}{\partial x_{m+1}}$ is defined by (4.0.3). We have

$$\int_{B_R(x_0)} \left\langle \frac{\partial v}{\partial x_{m+1}}(x', |x_{m+1}|), \phi \right\rangle \mathrm{d}x = \int_{B_R^+(x_0)} \left\langle \frac{\partial v}{\partial x_{m+1}}(x', x_{m+1}), \phi \right\rangle \mathrm{d}x + \int_{B_R^-(x_0)} \left\langle \frac{\partial v}{\partial x_{m+1}}(x', -x_{m+1}), \phi \right\rangle \mathrm{d}x. \tag{4.0.7}$$

Since ϕ vanishes on $\partial^+ B_R^+(x_0)$ and $Tv(x') = 0$ for $x' \in B_r^m(x_0)$, we integrate by parts to see that

$$\int_{B_R^+(x_0)} \left\langle \frac{\partial v}{\partial x_{m+1}}, \phi \right\rangle \mathrm{d}x = -\int_{B_R^+(x_0)} \left\langle \tilde{v}, \frac{\partial \phi}{\partial x_{m+1}} \right\rangle \mathrm{d}x. \tag{4.0.8}$$

Similarly, also using the change of variables $x_{m+1} \mapsto -x_{m+1}$, followed by an integration by parts and the same change of variables again, we find

$$\begin{aligned}\int_{B_R^-(x_0)} \left\langle \frac{\partial v}{\partial x_{m+1}}(x', -x_{m+1}), \phi \right\rangle \mathrm{d}x &= \int_{B_R^+(x_0)} \left\langle \frac{\partial v}{\partial x_{m+1}}, \phi(x', -x_{m+1}) \right\rangle \mathrm{d}x \\ &= \int_{B_R^+(x_0)} \left\langle v, \frac{\partial \phi}{\partial x_{m+1}}(x', -x_{m+1}) \right\rangle \mathrm{d}x \\ &= -\int_{B_R^-(x_0)} \left\langle \tilde{v}, \frac{\partial \phi}{\partial x_{m+1}} \right\rangle \mathrm{d}x. \qquad (4.0.9)\end{aligned}$$

The combination of (4.0.7), (4.0.8) and (4.0.9) shows that the weak partial derivative $\frac{\partial \tilde{v}}{\partial x_{m+1}}$ is defined by (4.0.3). Thus, since

$$\int_{B_R(x_0)} |x_{m+1}|^\beta |\nabla \tilde{v}|^2 \mathrm{d}x = 2 \int_{B_R^+(x_0)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \qquad (4.0.10)$$

and

$$\int_{B_R(x_0)} |x_{m+1}|^\beta |\tilde{v}|^2 \mathrm{d}x = 2 \int_{B_R^+(x_0)} x_{m+1}^\beta |v|^2 \mathrm{d}x, \qquad (4.0.11)$$

we deduce that $\tilde{v} \in W_\beta^{1,2}(B_R(x_0); \mathbb{R}^n)$.

Now we show that $\tilde{v}$ is a weak solution of (2.4.1) in $B_R(x_0)$. Let $\phi \in C_0^\infty(B_R(x_0); \mathbb{R}^n)$. We need to show that

$$\int_{B_R(x_0)} |x_{m+1}|^\beta \left\langle \nabla \tilde{v}, \nabla \phi \right\rangle \mathrm{d}x = 0.$$

To this end, consider $\chi \in C^\infty(\mathbb{R}; [0,1])$ with

$$\chi(s) = \begin{cases} 0 & s \in (-\infty, \frac{1}{2}] \\ 0 \le \chi(s) \le 1 & s \in (\frac{1}{2}, 1) \\ 1 & s \in [1, \infty) \end{cases}$$

Then, for $\delta > 0$, consider the symmetric functions $\chi_\delta(x_{m+1}) = \chi(\frac{|x_{m+1}|}{\delta}) \in C^\infty(\mathbb{R}; [0,1])$. The pointwise limit as $\delta \to 0^+$ of these functions is

$$\chi_0(s) = \begin{cases} 0 & x_{m+1} = 0 \\ 1 & x_{m+1} \in \mathbb{R}\backslash\{0\}. \end{cases}$$

Differentiating with respect to x_{m+1} we see that

$$(\chi_\delta)'(x_{m+1}) = 0$$

outside the set $[-\delta, -\frac{\delta}{2}] \cup [\frac{\delta}{2}, \delta]$ and furthermore,

$$(\chi_\delta)'(x_{m+1}) = \frac{1}{\delta}\mathrm{sgn}(x_{m+1})\chi'\left(\frac{|x_{m+1}|}{\delta}\right) \tag{4.0.12}$$

on this set.

Consider

$$\begin{aligned}\int_{B_R(x_0)} |x_{m+1}|^\beta \left\langle \nabla\tilde{v}, \nabla(\chi_{\frac{1}{k}}\phi)\right\rangle \mathrm{d}x = &\int_{B_R(x_0)} \chi_{\frac{1}{k}}|x_{m+1}|^\beta \left\langle \nabla\tilde{v}, \nabla\phi\right\rangle \mathrm{d}x \\ &+ \int_{B_R(x_0)} (\chi_{\frac{1}{k}})'|x_{m+1}|^\beta \left\langle \frac{\partial\tilde{v}}{\partial x_{m+1}}, \phi\right\rangle \mathrm{d}x,\end{aligned} \tag{4.0.13}$$

for $k \in \mathbb{N}$. We examine each of the terms in (4.0.13) in turn, with a view to taking the limit as $k \to \infty$.

First, we write

$$\begin{aligned}\int_{B_R(x_0)} |x_{m+1}|^\beta \left\langle \nabla\tilde{v}, \nabla(\chi_{\frac{1}{k}}\phi)\right\rangle \mathrm{d}x = &\int_{B_R^+(x_0)} x_{m+1}^\beta \left\langle \nabla\tilde{v}, \nabla(\chi_{\frac{1}{k}}\phi)\right\rangle \mathrm{d}x \\ &+ \int_{B_R^-(x_0)} (-x_{m+1})^\beta \left\langle \nabla\tilde{v}, \nabla(\chi_{\frac{1}{k}}\phi)\right\rangle \mathrm{d}x.\end{aligned} \tag{4.0.14}$$

As v is a weak solution of $\mathrm{div}(x_{m+1}^\beta \nabla v) = 0$ in $B_R^+(x_0)$ and $\chi_{\frac{1}{k}}\phi \in C_0^\infty(B_R^+(x_0); \mathbb{R}^n)$ for every k, it follows that

$$\int_{B_R^+(x_0)} x_{m+1}^\beta \left\langle \nabla\tilde{v}, \nabla(\chi_{\frac{1}{k}}\phi)\right\rangle \mathrm{d}x = 0. \tag{4.0.15}$$

Furthermore, we calculate

$$\begin{aligned}&\int_{B_R^-(x_0)} (-x_{m+1})^\beta \left\langle \nabla\tilde{v}, \nabla(\chi_{\frac{1}{k}}\phi)\right\rangle \mathrm{d}x \\ &= -\int_{B_R^-(x_0)} \chi_{\frac{1}{k}}(-x_{m+1})^\beta \left\langle \nabla' v(x', -x_{m+1}), \nabla'\phi\right\rangle \mathrm{d}x \\ &\quad + \int_{B_R^-(x_0)} \chi_{\frac{1}{k}}(-x_{m+1})^\beta \left\langle \frac{\partial v}{\partial x_{m+1}}(x', -x_{m+1}), \frac{\partial\phi}{\partial x_{m+1}}\right\rangle \mathrm{d}x \\ &\quad - \int_{B_R^-(x_0)} k\chi'(-kx_{m+1})(-x_{m+1})^\beta \left\langle \frac{\partial v}{\partial x_{m+1}}(x', -x_{m+1}), \phi\right\rangle \mathrm{d}x.\end{aligned} \tag{4.0.16}$$

Using the change of variables $x_{m+1} \mapsto -x_{m+1}$ combined with (4.0.12) we see that

$$\begin{aligned}\int_{B_R^-(x_0)} &\chi_{\frac{1}{k}}(-x_{m+1})^{\beta} \langle \nabla' v(x', -x_{m+1}), \nabla' \phi \rangle \, \mathrm{d}x \\ &= \int_{B_R^+(x_0)} \chi_{\frac{1}{k}} x_{m+1}^{\beta} \langle \nabla' v, \nabla' \phi(x', -x_{m+1}) \rangle \, \mathrm{d}x, \end{aligned} \tag{4.0.17}$$

$$\begin{aligned}\int_{B_R^-(x_0)} &\chi_{\frac{1}{k}}(-x_{m+1})^{\beta} \left\langle \frac{\partial v}{\partial x_{m+1}}(x', -x_{m+1}), \frac{\partial \phi}{\partial x_{m+1}} \right\rangle \mathrm{d}x \\ &= -\int_{B_R^+(x_0)} \chi_{\frac{1}{k}} x_{m+1}^{\beta} \left\langle \frac{\partial v}{\partial x_{m+1}}, \frac{\partial}{\partial x_{m+1}}(\phi(x', -x_{m+1})) \right\rangle \mathrm{d}x \end{aligned} \tag{4.0.18}$$

and

$$\begin{aligned} -\int_{B_R^-(x_0)} &k\chi'(-kx_{m+1})(-x_{m+1})^{\beta} \left\langle \frac{\partial v}{\partial x_{m+1}}(x', -x_{m+1}), \phi \right\rangle \mathrm{d}x \\ &= -\int_{B_R^+(x_0)} k\chi'(kx_{m+1}) x_{m+1}^{\beta} \left\langle \frac{\partial v}{\partial x_{m+1}}, \phi(x', -x_{m+1}) \right\rangle \mathrm{d}x. \end{aligned} \tag{4.0.19}$$

Hence, combining (4.0.16), (4.0.17), (4.0.18) and (4.0.19) we see that

$$\begin{aligned}\int_{B_R^-(x_0)} &(-x_{m+1})^{\beta} \left\langle \nabla \tilde{v}, \nabla(\chi_{\frac{1}{k}} \phi) \right\rangle \mathrm{d}x \\ &= -\int_{B_R^+(x_0)} x_{m+1}^{\beta} \left\langle \nabla v, \nabla(\chi_{\frac{1}{k}} \phi(x', -x_{m+1})) \right\rangle \mathrm{d}x. \end{aligned} \tag{4.0.20}$$

Now observe that the map $x \mapsto \phi(x', -x_{m+1}) \in C^{\infty}(B_R(x_0); \mathbb{R}^n)$ because $\phi \in C^{\infty}(B_R(x_0); \mathbb{R}^n)$. Thus, since v is a weak solution of $\operatorname{div}(x_{m+1}^{\beta} \nabla v) = 0$ in $B_R^+(x_0)$ and $x \mapsto \chi_{\frac{1}{k}} \phi(x', -x_{m+1}) \in C_0^{\infty}(B_R^+(x_0); \mathbb{R}^n)$, for every k we have

$$\int_{B_R^+(x_0)} x_{m+1}^{\beta} \left\langle \nabla v, \nabla(\chi_{\frac{1}{k}} \phi(x', -x_{m+1})) \right\rangle \mathrm{d}x = 0. \tag{4.0.21}$$

Together, (4.0.14), (4.0.15), (4.0.20) and (4.0.21) show that

$$\int_{B_R(x_0)} |x_{m+1}|^{\beta} \left\langle \nabla \tilde{v}, \nabla(\chi_{\frac{1}{k}} \phi) \right\rangle \mathrm{d}x = 0 \tag{4.0.22}$$

for every $k \in \mathbb{N}$.

Now we consider the other terms in (4.0.13). Since $\tilde{v}, \phi \in W_{\beta}^{1,2}(B_R(x_0); \mathbb{R}^n)$, we have $|\chi_{\frac{1}{k}} |x_{m+1}|^{\beta} \langle \nabla \tilde{v}, \nabla \phi \rangle| \leq C|x_{m+1}|^{\beta} |\nabla \tilde{v}||\nabla \phi| \in L^1(B_R(x_0); \mathbb{R}^n)$. Thus, Lebesgue's Dominated Convergence Theorem implies that

$$\lim_{k\to\infty}\int_{B_R(x_0)}\chi_{\frac{1}{k}}|x_{m+1}|^{\beta}\left\langle\nabla\tilde{v},\nabla\phi\right\rangle\mathrm{d}x=\int_{B_R(x_0)}|x_{m+1}|^{\beta}\left\langle\nabla\tilde{v},\nabla\phi\right\rangle\mathrm{d}x. \tag{4.0.23}$$

Lastly we consider the term $\int_{B_R(x_0)}(\chi_{\frac{1}{k}})'|x_{m+1}|^{\beta}\left\langle\frac{\partial\tilde{v}}{\partial x_{m+1}},\phi\right\rangle\mathrm{d}x$. Let $f_k(x')=\min\{\frac{1}{k},f(x')\}$ where $f(x')=(R^2-|x'-x_0|^2)^{\frac{1}{2}}$ for $x'\in B_R^m(x_0)$. Then, recalling (4.0.12) and the fact that $(\chi_{\frac{1}{k}})'(x_{m+1})=0$ when $x_{m+1}\notin[-\frac{1}{k},-\frac{1}{2k}]\cup[\frac{1}{2k},\frac{1}{k}]$, we have

$$\begin{aligned}&\int_{B_R(x_0)}(\chi_{\frac{1}{k}})'|x_{m+1}|^{\beta}\left\langle\frac{\partial\tilde{v}}{\partial x_{m+1}},\phi\right\rangle\mathrm{d}x\\&=\int_{B_R^m(x_0)}\int_0^{f_k}k\chi'(kx_{m+1})x_{m+1}^{\beta}\left\langle\frac{\partial v}{\partial x_{m+1}},\phi\right\rangle\mathrm{d}x\\&\quad-\int_{B_R^m(x_0)}\int_{-f_k}^{0}k\chi'(-kx_{m+1})(-x_{m+1})^{\beta}\left\langle\frac{\partial v}{\partial x_{m+1}}(x',-x_{m+1}),\phi\right\rangle\mathrm{d}x.\end{aligned}$$

Using the change of variables $x_{m+1}\mapsto -x_{m+1}$ we see that

$$\begin{aligned}&\int_{B_R(x_0)}(\chi_{\frac{1}{k}})'|x_{m+1}|^{\beta}\left\langle\frac{\partial\tilde{v}}{\partial x_{m+1}},\phi\right\rangle\mathrm{d}x\\&=\int_{B_R^m(x_0)}\int_0^{f_k}k\chi'(kx_{m+1})x_{m+1}^{\beta}\left\langle\frac{\partial v}{\partial x_{m+1}},\phi(x',x_{m+1})-\phi(x',-x_{m+1})\right\rangle\mathrm{d}x.\end{aligned} \tag{4.0.24}$$

Furthermore, we have $|\phi(x',x_{m+1})-\phi(x',-x_{m+1})|\le\frac{2}{k}||\nabla\phi||_{L^{\infty}(B_R(x_0);\mathbb{R}^{(m+1)n})}$ for every $x_{m+1}\in(0,f_k(x'))$ and every $x'\in B_R^m(x_0)$ as a result of the Mean Value Theorem. This fact, together with (4.0.24) yields

$$\begin{aligned}&\left|\int_{B_R(x_0)}(\chi_{\frac{1}{k}})'|x_{m+1}|^{\beta}\left\langle\frac{\partial\tilde{v}}{\partial x_{m+1}},\phi\right\rangle\mathrm{d}x\right|\\&\le C||\nabla\phi||_{L^{\infty}(B_R(x_0);\mathbb{R}^{(m+1)n})}\int_{B_R^m(x_0)}\int_0^{f_k}x_{m+1}^{\beta}\left|\frac{\partial v}{\partial x_{m+1}}\right|\mathrm{d}x.\end{aligned} \tag{4.0.25}$$

Since $v\in W_{\beta}^{1,2}(B_R^+(x_0);\mathbb{R}^n)$, we may apply Lebesgue's Dominated Convergence Theorem to see that

$$\int_{B_R^m(x_0)}\int_0^{f_k}x_{m+1}^{\beta}\left|\frac{\partial v}{\partial x_{m+1}}\right|\mathrm{d}x\to 0\text{ as }k\to\infty. \tag{4.0.26}$$

Consequently, in view of (4.0.25) and (4.0.26), we have

$$\lim_{k\to\infty}\int_{B_R(x_0)}(\chi_{\frac{1}{k}})'|x_{m+1}|^\beta\left\langle\frac{\partial\tilde{v}}{\partial x_{m+1}},\phi\right\rangle\mathrm{d}x=0. \tag{4.0.27}$$

Finally, we combine (4.0.13) with (4.0.22), (4.0.23) and (4.0.27) to see that

$$\int_{B_R(x_0)}|x_{m+1}|^\beta\langle\nabla\tilde{v},\nabla\phi\rangle\,\mathrm{d}x=\lim_{k\to\infty}\int_{B_R(x_0)}|x_{m+1}|^\beta\left\langle\nabla\tilde{v},\nabla(\chi_{\frac{1}{k}}\phi)\right\rangle\mathrm{d}x=0$$

as required. □

Proof of Lemma 4.0.0.2. First we show that $v(x)=\frac{1}{1-\beta}x_{m+1}^{1-\beta}$ is in $C^0(\overline{B_1^+(0)};\mathbb{R})\cap W_\beta^{1,2}(B_1^+(0);\mathbb{R})$ and is a weak solution of $\mathrm{div}(x_{m+1}^\beta\nabla v)=0$ in $B_1^+(0)$ with $v(x',0)=0$ for every $x'\in\mathbb{R}^m$.

Observe that v is smooth in $\mathbb{R}^{m+1}_+$ and continuous in $\overline{B_1^+(0)}$ by definition with $v(x',0)=0$ for all $x'\in\mathbb{R}^m$. Notice that

$$\frac{\partial v}{\partial x_i}=0\text{ for }i=1,\dots,m\quad\text{and}\quad\frac{\partial v}{\partial x_{m+1}}=x_{m+1}^{-\beta}\quad\text{and}\quad\frac{\partial^2 v}{\partial x_{m+1}^2}=-\beta x_{m+1}^{-\beta-1} \tag{4.0.28}$$

classically in $\mathbb{R}^{m+1}_+$. It follows that

$$\int_{B_1^+(0)}x_{m+1}^\beta|\nabla v|^2\,\mathrm{d}x=\int_{B_1^+(0)}x_{m+1}^\beta\left|\frac{\partial v}{\partial x_{m+1}}\right|^2\mathrm{d}x=\int_{B_1^+(0)}x_{m+1}^{-\beta}\mathrm{d}x=C$$

where $C=C(m,\beta)$. Hence, since v is continous, and therefore bounded, in $\overline{B_1^+(0)}$ and continuously differentiable in $B_1^+(0)$, it follows that $v\in W_\beta^{1,2}(B_1^+(0);\mathbb{R})$. Moreover, the continuity of v in $\overline{B_1^+(0)}$, together with the fact that $v(x',0)=0$ for all $x'\in\overline{\partial^0 B_1^+(0)}$, yields $Tv=0$ on $\partial^0B_1^+(0)$ for the trace operator T defined in Section 2.3.2. Lastly, we calculate that

$$\mathrm{div}(x_{m+1}^\beta\nabla v)=x_{m+1}^\beta\frac{\partial^2 v}{\partial x_{m+1}^2}+\beta x_{m+1}^{\beta-1}\frac{\partial v}{\partial x_{m+1}}=-\frac{\beta}{x_{m+1}}+\frac{\beta}{x_{m+1}}=0$$

classically in $B_1^+(0)$ and hence v is a weak solution of $\mathrm{div}(x_{m+1}^\beta\nabla v)=0$ on this domain.

We now apply Lemma 4.0.0.3 to see that the odd reflection

$$\tilde{v}(x',x_{m+1})=\begin{cases}\frac{1}{1-\beta}x_{m+1}^{1-\beta} & \text{if }x_{m+1}\geq 0\\ -\frac{1}{1-\beta}(-x_{m+1})^{1-\beta} & \text{if }x_{m+1}<0\end{cases}$$

weakly solves $\operatorname{div}(|x_{m+1}|^\beta \nabla\tilde{v}) = 0$ in $B_1(0)$ and satisfies $\tilde{v} \in W^{1,2}_\beta(B_1(0);\mathbb{R})$. The lemma also tells us that all weak partial derivatives of $\tilde{v}$ with respect to x_i with $i = 1,\dots,m$ are zero and that $\frac{\partial \tilde{v}}{\partial x_{m+1}} = |x_{m+1}|^{-\beta} \in L^2_\beta(B_1(0);\mathbb{R})$. We proceed to show that the monotonicity formula (4.0.1) is not valid for $\tilde{v}$.

For $0 < \rho < 1$ we calculate

$$\begin{aligned}\int_{B_\rho(0)} |x_{m+1}|^\beta |\nabla\tilde{v}|^2 \mathrm{d}x &= \int_{B_\rho(0)} |x_{m+1}|^\beta ||x_{m+1}|^{-\beta}|^2 \mathrm{d}x \\ &= \int_{B_\rho(0)} |x_{m+1}|^{-\beta} \mathrm{d}x \\ &= \int_0^\rho r^{m-\beta} \int_{\mathbb{S}^m} |\omega_{m+1}|^{-\beta} \mathrm{d}\omega \mathrm{d}r \\ &= \rho^{1+m-\beta} \frac{\int_{\mathbb{S}^m} |\omega_{m+1}|^{-\beta} \mathrm{d}\omega}{1+m-\beta} \end{aligned} \tag{4.0.29}$$

since $\int_{\mathbb{S}^m} |\omega_{m+1}|^{-\beta} \mathrm{d}\omega < \infty$. Thus we deduce that

$$r^{-(1+m+\beta)} \int_{B_r(0)} |x_{m+1}|^\beta |\nabla\tilde{v}|^2 \mathrm{d}x \leq R^{-(1+m+\beta)} \int_{B_R(0)} |x_{m+1}|^\beta |\nabla\tilde{v}|^2 \mathrm{d}x \tag{4.0.30}$$

if, and only if,

$$r^{-(1+m+\beta)} r^{1+m-\beta} \frac{\int_{\mathbb{S}^m} |\omega_{m+1}|^{-\beta} \mathrm{d}\omega}{1+m-\beta} \leq R^{-(1+m+\beta)} R^{1+m-\beta} \frac{\int_{\mathbb{S}^m} |\omega_{m+1}|^{-\beta} \mathrm{d}\omega}{1+m-\beta} \tag{4.0.31}$$

if, and only if,

$$r^{-2\beta} \leq R^{-2\beta}, \tag{4.0.32}$$

which is false if $0 < r < R < 1$ and $\beta \in (0,1)$. □

4.1 Monotonicity Formula for Even Solutions of the Linear Degenerate Equation

This section is dedicated to proving the following monotonicity formula.

Theorem 4.1.0.1. *Let $B_R(x_0) \subset \mathbb{R}^{m+1}$ with $(x_0)_{m+1} = 0$ and $R \leq 1$ and suppose $v \in W^{1,2}_\beta(B_R(x_0);\mathbb{R}^n)$ is a weak solution of $\operatorname{div}(|x_{m+1}|^\beta \nabla v) = 0$ in $B_R(x_0)$. If v is symmetric with respect to $\partial\mathbb{R}^{m+1}_+$ in $B_R(x_0)$ then*

$$s^{-(1+m+\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x \leq r^{-(1+m+\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x$$

for every $0 < s \leq r \leq R$.

Remark 4.1.0.1. Recall that a weak solution of $\operatorname{div}(|x_{m+1}|^{\beta}\nabla v) = 0$ in $B_R(x_0)$ is symmetric with respect to $\partial\mathbb{R}^{m+1}_{+}$ if, and only if, it is a weak solution of the Neumann-type problem (2.4.5) in $B^{+}_{R}(x_0)$ by Lemma 2.4.2.2. Hence the above theorem applies to solutions of this problem as well.

We will use the notation $\partial_i^k v = \frac{\partial^k v}{\partial x_i^k}$ for $i = 1, \ldots, m+1$ and $k \in \mathbb{N}$. Furthermore, we define $v^* := |x_{m+1}|^{\beta}\partial_{m+1}v$. This function is significant and will be integral to our proof of Theorem 4.1.0.1 because, as we will see in more detail later, it satisfies $\operatorname{div}(|x_{m+1}|^{-\beta}\nabla v^*) = 0$ when $\operatorname{div}(|x_{m+1}|^{\beta}\nabla v) = 0$ and v is symmetric with respect to $\partial\mathbb{R}^{m+1}_{+}$.

We proceed by proving the required monotonicity holds for the derivatives $\partial_i v$, where $i = 1, \ldots, m$, of a solution to $\operatorname{div}(|x_{m+1}|^{\beta}\nabla v) = 0$ in $B_R(x_0)$. Treatment of the derivatives $\partial_{m+1}v$ requires a slightly different approach depending on whether $\beta < 0$ or $\beta > 0$.

First, we consider the differentiability of solutions of $\operatorname{div}(|x_{m+1}|^{\beta}\nabla v) = 0$ in general. We will describe results on differentiability in the directions x_i for $i = 1, \ldots, m$. In turn, these results will provide the information we need about the derivatives in the x_{m+1} direction when the solutions are symmetric with respect to $\partial\mathbb{R}^{m+1}_{+}$.

Let $i = 1, \ldots, m$ and let $h \in \mathbb{R}$. Define the difference quotient of a map $v : \Omega \to \mathbb{R}^n$ by $\Delta_i^h v(x) = h^{-1}(v(x + he_i) - v(x))$ where e_i denotes the i-th basis vector in $\mathbb{R}^{m+1}$ and $\operatorname{dist}(x, \partial\Omega) < |h|$.

Lemma 4.1.0.1. *Let* $\Omega \subset \mathbb{R}^{m+1}$ *be open and* $v \in W^{1,2}_{\beta}(\Omega; \mathbb{R}^n)$. *Then for any* $i = 1, \ldots, m$ *we have* $\Delta_i^h v \in L^2_{\beta}(K; \mathbb{R}^n)$ *for any compact* $K \subset \Omega$*, provided* $|h| <$ *dist*$(K, \partial\Omega)$. *In particular,*

$$\int_K |x_{m+1}|^{\beta}|\Delta_i^h v|^2 \mathrm{d}x \leq \int_{\Omega} |x_{m+1}|^{\beta}\,|\partial_i v|^2\,\mathrm{d}x.$$

Proof. This proof follows the proof of Lemma 7.23 in [21]. We assume that $h \geq 0$, the argument for negative h is analogous. Let $v \in C^1(\Omega; \mathbb{R}^n) \cap W^{1,2}_{\beta}(\Omega; \mathbb{R}^n)$. Then for any $i = 1, \ldots, m$ we have

$$\Delta_i^h v(x) = \frac{1}{h}\int_0^h \partial_i v(x + te_i)\mathrm{d}t.$$

Hence

$$|\Delta_i^h v(x)|^2 = \left|\frac{1}{h}\int_0^h \partial_i v(x+te_i)\mathrm{d}t\right|^2 \leq \frac{1}{h}\int_0^h |\partial_i v(x+te_i)|^2\,\mathrm{d}t.$$

Thus, using the notation $K_h = \{x \in \mathbb{R}^{m+1} : \mathrm{dist}(x,K) \ \leq h\}$ and noting $K_h \subset \Omega$, by Fubini's Theorem and the compactness of K, for h with $|h| < \mathrm{dist}(K, \partial\Omega)$ we have

$$\begin{aligned}\int_K |x_{m+1}|^\beta |\Delta_i^h v|^2 \mathrm{d}x &\leq \int_K |x_{m+1}|^\beta \frac{1}{h}\int_0^h |\partial_i v(x+te_i)|^2\,\mathrm{d}t\mathrm{d}x \\ &\leq \frac{1}{h}\int_0^h \int_{K_h} |x_{m+1}|^\beta\,|\partial_i v|^2\,\mathrm{d}x\mathrm{d}t \\ &\leq \int_\Omega |x_{m+1}|^\beta\,|\partial_i v|^2\,\mathrm{d}x.\end{aligned}$$

We deduce the result for $v \in W_\beta^{1,2}(\Omega;\mathbb{R}^n)$ by approximation. $\square$

Next we prove (essentially) the reverse implication. Let $\Omega \subset \mathbb{R}^{m+1}$ be a domain.

Lemma 4.1.0.2. *Suppose $\Omega \subset \mathbb{R}^{m+1}$ is open and bounded and let $v \in L_\beta^2(\Omega;\mathbb{R}^n)$. For any $i = 1,\ldots,m$, suppose there exist constants $M > 0$ and $\tilde{h} > 0$ such that*

$$\int_K |x_{m+1}|^\beta |\Delta_i^h v|^2 \mathrm{d}x \leq M$$

for every $h \neq 0$ with $|h| < \tilde{h}$ and compact $K \subset \Omega$ with $\mathrm{dist}(K,\partial\Omega) > |h|$. Then the weak derivative $\partial_i v$ exists in Ω and satisfies

$$\int_\Omega |x_{m+1}|^\beta\,|\partial_i v|^2\,\mathrm{d}x \leq M.$$

Proof. First choose a sequence $(h_k)_{k\in\mathbb{N}}$ with $h_k \to 0$, discard h_k with $|h_k| \geq \tilde{h}$, re-index to $k \in \mathbb{N}$ and define $v_i^{h_k}(x) = \Delta_i^{h_k} v(x)$ when $x \in \Omega$ and $\mathrm{dist}(x,\partial\Omega) \geq 2|h_k|$ and $v_i^{h_k} = 0$ otherwise. As a consequence of the assumptions in the lemma, $\{v_i^{h_k}\}_{k\in\mathbb{N}}$ is a bounded sequence in the Hilbert space $L_\beta^2(\Omega;\mathbb{R}^n)$. Hence there is a subsequence, which we do not relabel, such that $h_k \to 0$ and $v_i^{h_k} \to \tilde{v}_i$ weakly in $L_\beta^2(\Omega;\mathbb{R}^n)$. Furthermore, this convergence, together with the weak lower semi-continuity of a Hilbert space norm, guarantees that $\int_\Omega x_{m+1}^\beta\,|\tilde{v}_i|^2\,\mathrm{d}x \leq M$. Analogous calculations to those in the proof of Lemmata 2.2.1.2, 2.2.1.3 and 2.2.1.4 imply that $L_\beta^2(\Omega;\mathbb{R}^n) \subset L^p(\Omega;\mathbb{R}^n)$ for some $p \in (1,2]$ depending on β. Hence, since every linear functional on $L^p(\Omega;\mathbb{R}^n)$ restricts to a linear functional on $L^2_\beta(\Omega;\mathbb{R}^n)$, we also have that $v_i^{h_k}$ converges to $\tilde{v}_i$ weakly in $L^p(\Omega;\mathbb{R}^n)$. Thus we deduce, as in the proof of Lemma 7.24 in [21], that $\tilde{v}_i$ is the weak derivative $\partial_i v$. $\square$

Now we consider the derivatives $\partial_i v$ of solutions to $\operatorname{div}(|x_{m+1}|^\beta \nabla v) = 0$ for $i = 1, \dots, m$.

Lemma 4.1.0.3. *Let $v \in W^{1,2}_\beta(B_R(x_0);\mathbb{R}^n)$ and suppose v is a weak solution of $\operatorname{div}(|x_{m+1}|^\beta \nabla v) = 0$ in $B_R(x_0)$. Then for every $r < R$ and $i = 1, \dots, m$, $\partial_i v \in W^{1,2}_\beta(B_r(x_0);\mathbb{R}^n)$ and $\partial_i v$ is a weak solution of $\operatorname{div}(|x_{m+1}|^\beta \nabla v) = 0$ in $B_r(x_0)$.*

Proof. Recall that v satisfies

$$\int_{B_R(x_0)} |x_{m+1}|^\beta \langle \nabla v, \nabla \phi \rangle \,\mathrm{d}x \tag{4.1.1}$$

for every $\phi \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$ by Remark 2.4.0.1. Let $r < R$ and choose $\eta \in C_0^\infty(B_R(x_0))$ with $\eta \equiv 1$ in $B_r(x_0)$, $\eta \equiv 0$ in $B_R(x_0) \backslash B_{r+\frac{R-r}{2}}(x_0)$, $0 \le \eta \le 1$ and $|\nabla \eta| \le \frac{C}{R-r}$. Let $\Delta_i^h v$ be the difference quotient of v for some $i = 1, \dots, m$ and suppose $|h| < \frac{R-r}{4}$. Then $\phi = -\Delta_i^{-h}(\eta^2 \Delta_i^h v) \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$. We substitute this ϕ into (4.1.1) and integrate by parts to see that

$$\int_{B_R(x_0)} \eta^2 |x_{m+1}|^\beta |\nabla \Delta_i^h v|^2 \mathrm{d}x = -\int_{B_R(x_0)} 2\eta |x_{m+1}|^\beta \left\langle \nabla \Delta_i^h v \cdot \nabla \eta, \Delta_i^h v \right\rangle \mathrm{d}x.$$

Since $|\nabla \eta| \le \frac{C}{R-r}$ we have

$$\int_{B_R(x_0)} \eta^2 |x_{m+1}|^\beta |\nabla \Delta_i^h v|^2 \mathrm{d}x \le \frac{C}{R-r} \int_{B_R(x_0)} \eta |x_{m+1}|^\beta |\nabla \Delta_i^h v| |\Delta_i^h v| \mathrm{d}x.$$

An application of Young's inequality, $ab \le \delta \frac{a^2}{2} + \delta^{-1} \frac{a^2}{2}$ for $a, b \ge 0$ and $\delta > 0$, and Lemma 4.1.0.1 yields

$$\begin{aligned}\int_{B_R(x_0)} \eta^2 |x_{m+1}|^\beta |\nabla \Delta_i^h v|^2 \mathrm{d}x \le{}& \frac{C}{R-r}\delta \int_{B_R(x_0)} \eta^2 |x_{m+1}|^\beta |\nabla \Delta_i^h v|^2 \mathrm{d}x \\ &+ \frac{C}{R-r}\delta^{-1} \int_{B_R(x_0)} |x_{m+1}|^\beta |\partial_i v|^2 \mathrm{d}x.\end{aligned}$$

Choosing $\delta = \frac{R-r}{2C}$ we deduce that

$$\int_{B_r(x_0)} |x_{m+1}|^\beta |\nabla \Delta_i^h v|^2 \mathrm{d}x \leq \int_{B_R(x_0)} \eta^2 |x_{m+1}|^\beta |\nabla \Delta_i^h v|^2 \mathrm{d}x$$
$$\leq \frac{C}{(R-r)^2} \int_{B_R(x_0)} |x_{m+1}|^\beta |\partial_i v|^2 \mathrm{d}x.$$

The right hand side above is independent of h and so by Lemma 4.1.0.2 the weak derivative $\nabla \partial_i v$ exists and is in $L^2_\beta(B_r(x_0); \mathbb{R}^{(m+1)n})$. Hence $\partial_i v \in W^{1,2}_\beta(B_r(x_0); \mathbb{R}^n)$ for every $r < R$. Finally, we integrate by parts in (4.1.1) to see that

$$0 = \int_{B_r(x_0)} |x_{m+1}|^\beta \langle \nabla v, \nabla \partial_i \phi \rangle \, \mathrm{d}x = -\int_{B_r(x_0)} |x_{m+1}|^\beta \langle \nabla \partial_i v, \nabla \phi \rangle \, \mathrm{d}x.$$

Hence $\partial_i v$ is a weak solution of $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$ in $B_r(x_0)$ for every $r < R$. □

We will need to consider the higher regularity of solutions of $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$ in balls centred on $\partial \mathbb{R}^{m+1}_+$. Solutions of this equation are smooth in such a ball away from $\partial \mathbb{R}^{m+1}_+$ and we require results dealing explicitly with regularity on $\partial \mathbb{R}^{m+1}_+$. Estimates for the Hölder norms of all derivatives $D^{\tilde{\alpha}} v$, where $\tilde{\alpha}$ is a multi-index with $(\tilde{\alpha})_{m+1} = 0$, are given in Proposition 2.1, Corollary 2.5 and Proposition 2.6 of [7], for example. These estimates can be derived directly, or by applying known theory for equations of the type we consider, see for instance Theorem 6.6 of [25]. We only need the property of continuity, rather than the explicit bounds obtained in the aforementioned literature. For the most part, this follows from the theory already described in this chapter.

Lemma 4.1.0.4. *Suppose $v \in W^{1,2}_\beta(B_R(x_0); \mathbb{R}^n)$ and assume v weakly satisifes div$(|x_{m+1}|^\beta \nabla v) = 0$ in $B_R(x_0)$. Then the derivatives $D^{\tilde{\alpha}} v$, where $\tilde{\alpha}$ is a multi-index with $(\tilde{\alpha})_{m+1} = 0$, are continuous in $B_R(x_0)$. If v is symmetric with respect to $\partial \mathbb{R}^{m+1}_+$ then so is $D^{\tilde{\alpha}} v$ whenever $\tilde{\alpha}$ is a multi-index with $(\tilde{\alpha})_{m+1} = 0$.*

Proof. We know that v is smooth in $B_R(x_0) \backslash \partial \mathbb{R}^{m+1}_+$ by linear elliptic regularity theory. Lemma 4.1.0.3 implies that $\partial_i v$ is a weak solution of $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$ in $B_r(x_0)$ for every $r < R$. Hence, using Lemma 2.4.3.1, we deduce that $\partial_i v$ is locally Hölder continuous, and therefore continuous, in $B_r(x_0)$ for every $r < R$. As a result $\partial_i v$ is continuous in $B_R(x_0)$. Applying this process now to the derivatives $\partial_i \partial_j v$ for $i, j = 1, \ldots, m$, we see that these derivatives are weak solutions of $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$ in $B_r(x_0)$ for every $r < R$ and are therefore continuous in $B_R(x_0)$. We can inductively repeat this process for $D^{\tilde{\alpha}} v$ where $\tilde{\alpha}$ is a multi-index with $(\tilde{\alpha})_{m+1} = 0$. Symmetry of $D^{\tilde{\alpha}} v$ with respect to $\partial \mathbb{R}^{m+1}_+$ follows from the corresponding symmetry of v. $\square$

Now we need to discuss regularity in the x_{m+1} direction. First, we record a consequence of a result of [4].

Lemma 4.1.0.5 (Consequence of [4] Lemma 4.5)**.** *Let $\beta \in (-1,1)$ and let $v \in L^\infty(B^+_{2r}(x_0)) \cap W^{1,2}_\beta(B^+_{2r}(x_0))$ be a weak solution the Neumann-type problem 2.4.5 in $B^+_{2r}(x_0)$. Then there is a $\gamma \in (0,1)$ depending on m, β such that $v, x^\beta_{m+1} \frac{\partial v}{\partial x_{m+1}} \in C^{0,\gamma}(\overline{B^+_r(x_0)})$.*

Together with our previous theory, this result implies the following.

Lemma 4.1.0.6. *Suppose $v \in W^{1,2}_\beta(B_R(x_0); \mathbb{R}^n)$ and assume v is a weak solution of $div(|x_{m+1}|^\beta \nabla v) = 0$ in $B_R(x_0)$ which is symmetric with respect to $\partial \mathbb{R}^{m+1}_+$. Then $v^* := |x_{m+1}|^\beta \partial_{m+1} v$ is continuous in $B_R(x_0)$, $v^*|_{\partial^0 B^+_R(x_0)} = 0$ and v^* is odd with respect to $\partial \mathbb{R}^{m+1}_+$, that is $v^*(x', x_{m+1}) = -v^*(x', -x_{m+1})$ for every $(x', x_{m+1}) \in B_R(x_0)$. Furthermore, $v^* \in W^{1,2}_{-\beta}(B_r(x_0); \mathbb{R}^n)$ and satisfies*

$$div(|x_{m+1}|^{-\beta} \nabla v^*) = 0$$

weakly in $B_r(x_0)$ for every $r < R$.

Proof. If v is as stated in the assumptions of the lemma then elliptic regularity theory shows that v^* is smooth in $B_R(x_0) \backslash \partial \mathbb{R}^{m+1}_+$ [21]. Fix $y \in B_R(x_0)$ with $y_{m+1} = 0$ and choose $s > 0$ such that $\overline{B_s(y)} \subset B_R(x_0)$. Lemma 2.4.3.1 implies that v is continuous, and hence bounded, in $B_r(x_0)$ for every $r < R$. Hence choosing $r < R$ with $\overline{B_s(y)} \subset B_r(x_0)$ we see that v is continuous and bounded in $\overline{B_s(y)}$. Furthermore, since v is symmetric with respect to $\partial \mathbb{R}^{m+1}_+$, it follows from Lemma 2.4.2.2 that v is a weak solution of the Neumann type problem (2.4.5) in $B^+_R(x_0)$ and hence in $B^+_s(y)$. Thus we may apply Lemma 4.1.0.5 to see that v^* is continuous in $\overline{B^+_{\frac{s}{2}}(y)}$. This holds for every $y \in B_R(x_0)$ with $y_{m+1} = 0$ and so v^* is continuous in $B^+_R(x_0) \cup \partial^0 B^+_R(x_0)$. As a result, using the fact that v is a weak solution of the Neumann type problem (2.4.5) we see that $v^*(x', 0) = 0$ for every $(x', 0) \in \partial^0 B^+_R(x_0)$.

For every $x = (x', x_{m+1}) \in B_R(x_0) \backslash \partial \mathbb{R}^{m+1}_+$, using the symmetry of v with respect to $\partial \mathbb{R}^{m+1}_+$, we calculate $\partial_{m+1} v(x', x_{m+1}) = -\partial_{m+1} v(x', -x_{m+1})$. Consequently, $v^*(x', x_{m+1}) = -v^*(x', -x_{m+1})$ for every $(x', x_{m+1}) \in B_R(x_0) \backslash \partial \mathbb{R}^{m+1}_+$. It follows that $\lim_{x_{m+1} \to 0^-} v^*(x', x_{m+1}) = -\lim_{x_{m+1} \to 0^+} v^*(x', x_{m+1}) = 0$ for every $(x', 0) \in \partial^0 B^+_R(x_0)$. Thus we deduce that v^* is continuous in $B_R(x_0)$ and $v^*|_{\partial^0 B^+_R(x_0)} = 0$.

Next we show that $v^* \in W^{1,2}_{-\beta}(B_r(x_0);\mathbb{R}^n)$ for every $r < R$. To see this observe that Lemma 4.1.0.3 implies $\partial_i v \in W^{1,2}_{\beta}(B_r(x_0);\mathbb{R}^n)$ for every $r < R$ and $i = 1, \ldots, m$. It follows that

$$\int_{B_r(x_0)} |x_{m+1}|^{-\beta}|\partial_i v^*|^2 \mathrm{d}x = \int_{B_r(x_0)} |x_{m+1}|^{\beta}|\partial_i\partial_{m+1}|^2 \mathrm{d}x < \infty \tag{4.1.2}$$

for $i = 1, \ldots, m$. We also note that $\Delta' v \in W^{1,2}_{\beta}(B_r(x_0);\mathbb{R}^n)$ by Lemma 4.1.0.3, where Δ' is the Laplace operator with respect to the variables $x_1, \ldots, x_m$. Furthermore, as v solves $\operatorname{div}(|x_{m+1}|^{\beta}\nabla v) = 0$ classically in $B_R(x_0)\backslash\partial\mathbb{R}^{m+1}_+$, we have $|x_{m+1}|^{-\beta}\partial_{m+1}v^* = -\Delta' v \in W^{1,2}_{\beta}(B_r(x_0);\mathbb{R}^n)$ for every $r < R$. Hence

$$\begin{aligned}\int_{B_r(x_0)} |x_{m+1}|^{-\beta}|\partial_{m+1}v^*|^2 \mathrm{d}x &= \int_{B_r(x_0)} |x_{m+1}|^{\beta}||x_{m+1}|^{-\beta}\partial_{m+1}v^*|^2 \mathrm{d}x \\ &= \int_{B_r(x_0)} |x_{m+1}|^{\beta}|\Delta' v|^2 \mathrm{d}x < \infty. \end{aligned} \tag{4.1.3}$$

The combination of (4.1.2) and (4.1.3) implies $\int_{B_r(x_0)} |x_{m+1}|^{-\beta}|\nabla v^*|^2 \mathrm{d}x < \infty$ for every $r < R$ and since

$$\int_{B_r(x_0)} |x_{m+1}|^{-\beta}|v^*|^2 \mathrm{d}x = \int_{B_r(x_0)} |x_{m+1}|^{\beta}|\partial_{m+1}v|^2 \mathrm{d}x < \infty, \tag{4.1.4}$$

we deduce that $v^* \in W^{1,2}_{-\beta}(B_r(x_0);\mathbb{R}^n)$.

Finally we show that v^* is a weak solution of $\operatorname{div}(|x_{m+1}|^{-\beta}\nabla v^*) = 0$ in $B_r(x_0)$ for every $r < R$. In analogy with the calculations in [5] we calculate

$$\operatorname{div}(x_{m+1}^{-\beta}\nabla v^*) = x_{m+1}^{-\beta}\partial_{m+1}(\operatorname{div}(x_{m+1}^{\beta}\nabla v)) = 0 \tag{4.1.5}$$

in $B^+_R(x_0)$ classically. Hence, v^* is continuous in $\overline{B^+_r(x_0)}$ for every $r < R$ and weakly satisfies $\operatorname{div}(x_{m+1}^{-\beta}\nabla(v^*|_{\overline{B^+_r(x_0)}})) = 0$ on these sets. Furthermore, $v^*|_{\partial^0 B^+_r(x_0)} = 0$ so Lemma 4.0.0.3 thus implies that the odd reflection of v^* in $\partial\mathbb{R}^{m+1}_+$ is a weak solution of $\operatorname{div}(|x_{m+1}|^{-\beta}\nabla v^*) = 0$ in $B_r(x_0)$. However, the odd reflection of $v^*|_{\overline{B^+_r(x_0)}}$ coincides with the way v^* is already defined. This holds for all $r < R$ which concludes the proof. $\square$

With the preceding regularity theory in hand, we are ready to commence the proof of Theorem 4.1.0.1. As mentioned previously, we treat the derivatives x_i for $i = 1, \ldots, m$ first and then consider those with respect to x_{m+1}. We begin with the following preliminary lemma.

Lemma 4.1.0.7. *Suppose that $v \in W^{1,2}_\beta(B_R(x_0))$ and that*

$$\int_{\partial B_\rho(x_0)} \nu \cdot |x_{m+1}|^\beta \nabla v \mathrm{d}\sigma(x) \geq 0$$

for almost every $\rho \in (0, R)$, where ν is the outward pointing unit normal on $\partial B_\rho(x_0)$, then

$$s^{-(1+m+\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta v \mathrm{d}x \leq r^{-(1+m+\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta v \mathrm{d}x \tag{4.1.6}$$

for every $0 < s \leq r \leq R$.

Proof. This proof follows the proof of Theorem 2.1 in Section 2 of [21] and the proof of Proposition 2.2 in Section III of [20]. Suppose v satisfies the assumptions of the lemma. For almost every $0 < s \leq r < R$ the following holds. Using Fubini's Theorem, we calculate

$$\begin{aligned}
&r^{-(m+\beta)} \int_{\partial B_r(x_0)} |x_{m+1}|^\beta v \mathrm{d}\sigma(x) - s^{-(m+\beta)} \int_{\partial B_s(x_0)} |x_{m+1}|^\beta v \mathrm{d}\sigma(x) \\
&= \int_{\partial B_1(0)} |\omega_{m+1}|^\beta (v(r\omega + x_0) - v(s\omega + x_0)) \mathrm{d}\omega \\
&= \int_{\partial B_1(0)} |\omega_{m+1}|^\beta \int_s^r \frac{\partial}{\partial t} v(t\omega + x_0) \mathrm{d}t \mathrm{d}\omega \\
&= \int_s^r \int_{\partial B_1(0)} |\omega_{m+1}|^\beta \omega \cdot \nabla v(t\omega + x_0) \mathrm{d}\omega \mathrm{d}t \\
&= \int_s^r t^{-(m+\beta)} \int_{\partial B_t(x_0)} |x_{m+1}|^\beta \nu \cdot \nabla v \mathrm{d}\sigma(x) \mathrm{d}t \\
&\geq 0.
\end{aligned} \tag{4.1.7}$$

Define the function $f(r) = \int_{B_r(x_0)} |x_{m+1}|^\beta v \mathrm{d}x$ for $0 < r \leq R$ and observe that $f(r)$ is an absolutely continuous function of r with $f'(r) = \int_{\partial B_r(x_0)} |x_{m+1}|^\beta v \mathrm{d}\sigma(x)$. Using (4.1.7) we calculate

$$\begin{aligned}
f(r) = \int_0^r f'(\rho) \mathrm{d}\rho &= \int_0^r \rho^{m+\beta} \rho^{-(m+\beta)} f'(\rho) \mathrm{d}\rho \\
&\leq \int_0^r \rho^{m+\beta} r^{-(m+\beta)} f'(r) \mathrm{d}\rho \\
&= \frac{r}{1+m+\beta} f'(r)
\end{aligned} \tag{4.1.8}$$

for $0 < r < R$. Multiplying the above inequality by $r^{-(1+m+\beta)}$ and rearranging yields

$$0 \leq r^{-(1+m+\beta)} f'(r) - (1+m+\beta) r^{-(2+m+\beta)} f(r). \tag{4.1.9}$$

As a result, we calculate

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^{-(1+m+\beta)} f(r)) = r^{-(1+m+\beta)} f'(r) - (1+m+\beta) r^{-(2+m+\beta)} f(r) \geq 0. \tag{4.1.10}$$

Integrating between $s \leq r \leq R$ completes the proof. $\square$

Now we consider the monotonicity formula for derivatives with respect to the variables x_i for $i = 1, \ldots, m$.

Lemma 4.1.0.8. *Let $v \in W^{1,2}_\beta(B_R(x_0); \mathbb{R}^n)$ and suppose that v is a weak solution of $\mathrm{div}(|x_{m+1}|^\beta \nabla v) = 0$ in $B_R(x_0)$ which is symmetric with respect to $\partial \mathbb{R}^{m+1}_+$. Let $\tilde{\alpha}$ be a multi-index with $\tilde{\alpha}_{m+1} = 0$. Then for every $0 < s \leq r < R$ we have*

$$s^{-(1+m+\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta |D^{\tilde{\alpha}} v|^2 \mathrm{d}x \leq r^{-(1+m+\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |D^{\tilde{\alpha}} v|^2 \mathrm{d}x.$$

Proof. First note that v is smooth in $B_R(x_0) \backslash \partial \mathbb{R}^{m+1}_+$ and $\mathrm{div}(|x_{m+1}|^\beta \nabla D^{\tilde{\alpha}} v) = 0$ classically in this set for any multi-index with $\tilde{\alpha}_{m+1} = 0$. Fix $R > r > \varepsilon > 0$. Define $B^\varepsilon_r(x_0) = B_r(x_0) \cap \{x \in \mathbb{R}^{m+1} : |x_{m+1}| > \varepsilon\}$. Observe that in $B^\varepsilon_r(x_0)$ we have

$$\mathrm{div}(|x_{m+1}|^\beta \nabla |D^{\tilde{\alpha}} v|^2) = 2|x_{m+1}|^\beta |\nabla D^{\tilde{\alpha}} v|^2 + 2\langle D^{\tilde{\alpha}} v, \mathrm{div}(|x_{m+1}|^\beta \nabla D^{\tilde{\alpha}} v)\rangle \geq 0$$

classically. Furthermore, notice that for any $r > \varepsilon > 0$ the domain $B^\varepsilon_r(x_0)$ is Lipschitz. Hence the Divergence Theorem may be applied on this set. We calculate

$$\begin{aligned} 0 &\leq \int_{B^\varepsilon_r(x_0)} \mathrm{div}(|x_{m+1}|^\beta \nabla |D^{\tilde{\alpha}} v|^2) \mathrm{d}x \\ &= \int_{\partial B_r(x_0)} \mathbb{1}_{\overline{B^\varepsilon_r(x_0)}} |x_{m+1}|^\beta \nu \cdot \nabla |D^{\tilde{\alpha}} v|^2 \mathrm{d}\sigma(x) \\ &\quad - \int_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \varepsilon^\beta e_{m+1} \cdot \nabla (|D^{\tilde{\alpha}} v|^2(x', \varepsilon) - |D^{\tilde{\alpha}} v|^2(x', -\varepsilon)) \mathrm{d}x' \end{aligned} \tag{4.1.11}$$

where $\mathbb{1}_{\overline{B^\varepsilon_r(x_0)}}$ is the indicator function of $\overline{B^\varepsilon_r(x_0)}$ and ν is the outward unit normal on $\partial B_r(x_0)$. We consider the terms on the right hand side above separately with a view to taking the limit as $\varepsilon \to 0^+$.

To facilitate this we recall the necessary regularity results for v. For every $r < R$, each $D^{\tilde{\alpha}}v \in W^{1,2}_\beta(B_r(x_0);\mathbb{R}^n)$ and is a weak solution of $\operatorname{div}(|x_{m+1}|^\beta \nabla D^{\tilde{\alpha}}v) = 0$ in $B_r(x_0)$ by Lemma 4.1.0.3. Moreover, each $D^{\tilde{\alpha}}v$ is continuous in $B_R(x_0)$ and symmetric with respect to $\partial\mathbb{R}^{m+1}_+$ by Lemma 4.1.0.4. Hence, Lemma 4.1.0.6 thus implies $(D^{\tilde{\alpha}}v)^* = |x_{m+1}|^\beta \partial_{m+1} D^{\tilde{\alpha}}v$ is continuous in $B_R(x_0)$ with $(D^{\tilde{\alpha}}v)^*|_{\partial^0 B^+_R(x_0)} = 0$. Since $(D^{\tilde{\alpha}}v)^* = |x_{m+1}|^\beta \partial_{m+1} D^{\tilde{\alpha}}v$ and $\partial_i D^{\tilde{\alpha}}v$, where $i = 1, \ldots, m$, are continuous, they are bounded in $\overline{B_r(x_0)}$ for every $r < R$. Consequently we calculate

$$\begin{aligned}
\left||x_{m+1}|^\beta \nu \cdot \nabla |D^{\tilde{\alpha}}v|^2\right| &\leq 2 \left||x_{m+1}|^\beta \left\langle \nu_{m+1}\partial_{m+1} D^{\tilde{\alpha}}v, D^{\tilde{\alpha}}v \right\rangle\right| \\
&\quad + 2 \left| \sum_{i=1}^m |x_{m+1}|^\beta \left\langle \nu_i \partial_i D^{\tilde{\alpha}}v, D^{\tilde{\alpha}}v \right\rangle \right| \\
&\leq C(1 + |x_{m+1}|^\beta).
\end{aligned}$$

It follows that

$$\left|\mathbb{1}_{\overline{B^\varepsilon_r(x_0)}} |x_{m+1}|^\beta \nu \cdot \nabla |D^{\tilde{\alpha}}v|^2\right| \leq C(1 + |x_{m+1}|^\beta)$$

for every $\varepsilon > 0$. Therefore, applying Lebesgue's Dominated Convergence Theorem, we see that

$$\lim_{\varepsilon \to 0^+} \int_{\partial B_r(x_0)} \mathbb{1}_{\overline{B^\varepsilon_r(x_0)}} |x_{m+1}|^\beta \nu \cdot \nabla |D^{\tilde{\alpha}}v|^2 \mathrm{d}\sigma(x) = \int_{\partial B_r(x_0)} |x_{m+1}|^\beta \nu \cdot \nabla |D^{\tilde{\alpha}}v|^2 \mathrm{d}\sigma(x). \tag{4.1.12}$$

Now we deal with the other term on the right hand side of (4.1.11). Since $\partial_{m+1} D^{\tilde{\alpha}}v(x', \varepsilon) = -\partial_{m+1} D^{\tilde{\alpha}}v(x', -\varepsilon)$ and $D^{\tilde{\alpha}}v(x', \varepsilon) = D^{\tilde{\alpha}}v(x', -\varepsilon)$, we calculate

$$\begin{aligned}
&e_{m+1} \cdot \nabla(|D^{\tilde{\alpha}}v|^2(x', \varepsilon) - |D^{\tilde{\alpha}}v|^2(x', -\varepsilon)) \\
&= 2\left\langle \partial_{m+1} D^{\tilde{\alpha}}v(x', \varepsilon), D^{\tilde{\alpha}}v(x', \varepsilon)\right\rangle - 2\left\langle \partial_{m+1} D^{\tilde{\alpha}}v(x', -\varepsilon), D^{\tilde{\alpha}}v(x', -\varepsilon)\right\rangle \\
&= 4\left\langle \partial_{m+1} D^{\tilde{\alpha}}v(x', \varepsilon), D^{\tilde{\alpha}}v(x', \varepsilon)\right\rangle.
\end{aligned}$$

Moreover, we note that for any $r < R$, $D^{\tilde{\alpha}}v$ is uniformly bounded on $\overline{B_r(x_0)}$ and $\mathbb{1}_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \varepsilon^\beta \partial_{m+1} D^{\tilde{\alpha}}v(x', \varepsilon) = \mathbb{1}_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} (D^{\tilde{\alpha}}v)^*(x', \varepsilon) \to 0$ uniformly in $\overline{B^m_r(x_0)}$ as $\varepsilon \to 0^+$. We deduce that $\mathbb{1}_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \varepsilon^\beta \left\langle \partial_{m+1} D^{\tilde{\alpha}}v(x', \varepsilon), D^{\tilde{\alpha}}v(x', \varepsilon)\right\rangle \to$

0 uniformly as $\varepsilon \to 0^+$. Hence

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\int_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \varepsilon^\beta e_{m+1}\cdot\nabla(|D^{\tilde\alpha}v|^2(x',\varepsilon)-|D^{\tilde\alpha}v|^2(x',-\varepsilon))\mathrm{d}x' \\
&= 4\lim_{\varepsilon\to 0^+}\int_{B^m_r(x_0)} \mathbb{1}_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)}\varepsilon^\beta\left\langle\partial_{m+1}D^{\tilde\alpha}v(x',\varepsilon),D^{\tilde\alpha}v(x',\varepsilon)\right\rangle\mathrm{d}x' = 0. \quad (4.1.13)
\end{aligned}$$

We combine (4.1.11), (4.1.12) and (4.1.13) to see that

$$0\le\lim_{\varepsilon\to 0^+}\int_{B^\varepsilon_r(x_0)}\mathrm{div}(|x_{m+1}|^\beta\nabla|D^{\tilde\alpha}v|^2)\mathrm{d}x=\int_{\partial B_r(x_0)}|x_{m+1}|^\beta\nu\cdot\nabla|D^{\tilde\alpha}v|^2\mathrm{d}\sigma(x). \quad (4.1.14)$$

This holds for every $r<R$. Applying Lemma 4.1.0.7 concludes the proof. □

Now we consider the monotonicity of derivatives with respect to x_{m+1}. The following lemma holds for all $\beta\in(-1,1)$ but, as we will see shortly, only yields sufficient information to prove Theorem 4.1.0.1 (when combined with the preceding Lemma) when $\beta\in(-1,0)$.

Lemma 4.1.0.9. *Suppose v is a weak solution of div$(|x_{m+1}|^\beta\nabla v)=0$ in $B_R(x_0)$ which is symmetric with respect to $\partial\mathbb{R}^{m+1}_+$. Let $\tilde\alpha$ be a multi-index with $\tilde\alpha_{m+1}=0$. Then for every $0<s\le r<R$ we have*

$$s^{-(1+m-\beta)}\int_{B_s(x_0)}|x_{m+1}|^\beta|\partial_{m+1}D^{\tilde\alpha}v|^2\mathrm{d}x\le r^{-(1+m-\beta)}\int_{B_r(x_0)}|x_{m+1}|^\beta|\partial_{m+1}D^{\tilde\alpha}v|^2\mathrm{d}x.$$

Proof. The proof of this lemma is similar to the proof of Lemma 4.1.0.8. Recall that v is smooth in $B_R(x_0)\backslash\partial\mathbb{R}^{m+1}_+$. It follows that $\mathrm{div}(|x_{m+1}|^{-\beta}\nabla(D^{\tilde\alpha}v)^*)=0$ classically in $B_R(x_0)\backslash\partial\mathbb{R}^{m+1}_+$. Fix $R>r>\varepsilon>0$. Define $B^\varepsilon_r(x_0)=B_r(x_0)\cap\{x\in\mathbb{R}^{m+1}:|x_{m+1}|>\varepsilon\}$. Observe that in $B^\varepsilon_r(x_0)$ we have

$$\begin{aligned}
\mathrm{div}(|x_{m+1}|^{-\beta}\nabla|(D^{\tilde\alpha}v)^*|^2) &= 2|x_{m+1}|^\beta|\nabla(D^{\tilde\alpha}v)^*|^2 \\
&\quad + 2\langle(D^{\tilde\alpha}v)^*,\mathrm{div}(|x_{m+1}|^{-\beta}\nabla(D^{\tilde\alpha}v)^*)\rangle\ge 0
\end{aligned}$$

classically. Furthermore, notice that for any $r>\varepsilon>0$ the domain $B^\varepsilon_r(x_0)$ is Lipschitz. Hence the Divergence Theorem may be applied on this set. We calculate

$$
\begin{aligned}
0 &\le \int_{B_r^\varepsilon(x_0)} \operatorname{div}(|x_{m+1}|^{-\beta}\nabla|(D^{\tilde{\alpha}}v)^*|^2)\mathrm{d}x \\
&= \int_{\partial B_r(x_0)} \mathbb{1}_{\overline{B_r^\varepsilon(x_0)}}|x_{m+1}|^{-\beta}\nu\cdot\nabla|(D^{\tilde{\alpha}}v)^*|^2\mathrm{d}\sigma(x) \\
&\quad - \int_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \varepsilon^{-\beta}e_{m+1}\cdot\nabla(|(D^{\tilde{\alpha}}v)^*|^2(x',\varepsilon)-|(D^{\tilde{\alpha}}v)^*|^2(x',-\varepsilon))\mathrm{d}x' \quad (4.1.15)
\end{aligned}
$$

where $\mathbb{1}_{\overline{B_r^\varepsilon(x_0)}}$ is the indicator function of $\overline{B_r^\varepsilon(x_0)}$ and ν is the outward unit normal on $\partial B_r(x_0)$. We consider the terms on the right hand side above separately with a view to taking the limit as $\varepsilon \to 0^+$.

We recall the regularity theory, for v and its derivatives, which is required to facilitate this procedure. For every $r < R$, each $D^{\tilde{\alpha}}v \in W^{1,2}_\beta(B_r(x_0);\mathbb{R}^n)$ and is a weak solution of $\operatorname{div}(|x_{m+1}|^\beta\nabla D^{\tilde{\alpha}}v) = 0$ in $B_r(x_0)$ by Lemma 4.1.0.3. Furthermore, each $D^{\tilde{\alpha}}v$ is continuous in $B_R(x_0)$ and symmetric with respect to $\partial\mathbb{R}^{m+1}_+$ by Lemma 4.1.0.4. We conclude, using Lemma 4.1.0.6, that $(D^{\tilde{\alpha}}v)^* = |x_{m+1}|^\beta\partial_{m+1}D^{\tilde{\alpha}}v$ is continuous in $B_R(x_0)$ with $(D^{\tilde{\alpha}}v)^*|_{\partial^0 B_R^+(x_0)} = 0$. We further note that Lemma 4.1.0.6 implies that for every $r < R$, we have $(D^{\tilde{\alpha}}v)^* \in W^{1,2}_{-\beta}(B_r(x_0);\mathbb{R}^n)$ and $(D^{\tilde{\alpha}}v)^*$ is a weak solution of $\operatorname{div}(|x_{m+1}|^{-\beta}\nabla(D^{\tilde{\alpha}}v)^*) = 0$ in $B_r(x_0)$. Another application of Lemma 4.1.0.4 implies that for every $r < R$ and $i = 1,\dots,m$, the derivatives $\partial_i(D^{\tilde{\alpha}}v)^*$ are continuous in $B_r(x_0)$.

Using the preceding observations we see that $|x_{m+1}|^{-\beta}\partial_{m+1}(D^{\tilde{\alpha}}v)^* = -\Delta' D^{\tilde{\alpha}}v$ is continuous in $B_R(x_0)$ and bounded in $\overline{B_r(x_0)}$ for every $r < R$. Furthermore, the continuity of $(D^{\tilde{\alpha}}v)^* = |x_{m+1}|^\beta\partial_{m+1}D^{\tilde{\alpha}}v$ and $\partial_i(D^{\tilde{\alpha}}v)^*$, where $i = 1,\dots,m$, yields their boundedness in $\overline{B_r(x_0)}$. Hence, we have

$$
\begin{aligned}
||x_{m+1}|^{-\beta}\nu\cdot\nabla|(D^{\tilde{\alpha}}v)^*|^2| &\le 2\left||x_{m+1}|^{-\beta}\left\langle \nu_{m+1}\partial_{m+1}(D^{\tilde{\alpha}}v)^*,(D^{\tilde{\alpha}}v)^*\right\rangle\right| \\
&\quad + 2\left|\sum_{i=1}^m |x_{m+1}|^{-\beta}\left\langle \nu_i\partial_i(D^{\tilde{\alpha}}v)^*,(D^{\tilde{\alpha}}v)^*\right\rangle\right| \\
&\le C(1+|x_{m+1}|^{-\beta}).
\end{aligned}
$$

It follows that

$$
|\mathbb{1}_{\overline{B_r^\varepsilon(x_0)}}|x_{m+1}|^{-\beta}\nu\cdot\nabla|(D^{\tilde{\alpha}}v)^*|^2| \le C(1+|x_{m+1}|^{-\beta})
$$

for every $\varepsilon \in (0,r)$. Therefore, applying Lebesgue's Dominated Convergence

Theorem, we see that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\int_{\partial B_r(x_0)} \mathbb{1}_{\overline{B_r^\varepsilon(x_0)}}|x_{m+1}|^{-\beta}\nu\cdot\nabla|(D^{\tilde{\alpha}}v)^*|^2\mathrm{d}\sigma(x)\\
&=\int_{\partial B_r(x_0)}|x_{m+1}|^{-\beta}\nu\cdot\nabla|(D^{\tilde{\alpha}}v)^*|^2\mathrm{d}\sigma(x). \qquad (4.1.16)
\end{aligned}$$

Now we deal with the other term on the right hand side of (4.1.15). We have $\partial_{m+1}(D^{\tilde{\alpha}}v)^*(x',\varepsilon) = \partial_{m+1}(D^{\tilde{\alpha}}v)^*(x',-\varepsilon)$ and $(D^{\tilde{\alpha}}v)^*(x',\varepsilon) = -(D^{\tilde{\alpha}}v)^*(x',-\varepsilon)$. Thus

$$\begin{aligned}
&e_{m+1}\cdot\nabla(|(D^{\tilde{\alpha}}v)^*|^2(x',\varepsilon)-|(D^{\tilde{\alpha}}v)^*|^2(x',-\varepsilon))\\
&=2\left\langle\partial_{m+1}(D^{\tilde{\alpha}}v)^*(x',\varepsilon),(D^{\tilde{\alpha}}v)^*(x',\varepsilon)\right\rangle-2\left\langle\partial_{m+1}(D^{\tilde{\alpha}}v)^*(x',-\varepsilon),(D^{\tilde{\alpha}}v)^*(x',-\varepsilon)\right\rangle\\
&=4\left\langle\partial_{m+1}(D^{\tilde{\alpha}}v)^*(x',\varepsilon),(D^{\tilde{\alpha}}v)^*(x',\varepsilon)\right\rangle.
\end{aligned}$$

We observe that $\mathbb{1}_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)}\varepsilon^\beta\partial_{m+1}D^{\tilde{\alpha}}v(x',\varepsilon)=\mathbb{1}_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)}(D^{\tilde{\alpha}}v)^*(x',\varepsilon)\to 0$ uniformly in $\overline{B_r^m(x_0)}$ as $\varepsilon\to 0^+$ and $|x_{m+1}|^{-\beta}\partial_{m+1}(D^{\tilde{\alpha}}v)^* = -\Delta' D^{\tilde{\alpha}}v$ is uniformly bounded $\overline{B_r(x_0)}$. Thus $\mathbb{1}_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)}\varepsilon^{-\beta}\left\langle\partial_{m+1}(D^{\tilde{\alpha}}v)^*(x',\varepsilon),(D^{\tilde{\alpha}}v)^*(x',\varepsilon)\right\rangle\to 0$ uniformly as $\varepsilon\to 0^+$. Hence

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\int_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)}\varepsilon^{-\beta}e_{m+1}\cdot\nabla(|(D^{\tilde{\alpha}}v)^*|^2(x',\varepsilon)-|(D^{\tilde{\alpha}}v)^*|^2(x',-\varepsilon))\mathrm{d}x'\\
&=4\lim_{\varepsilon\to 0^+}\int_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)}\varepsilon^{-\beta}\left\langle\partial_{m+1}(D^{\tilde{\alpha}}v)^*(x',\varepsilon),(D^{\tilde{\alpha}}v)^*(x',\varepsilon)\right\rangle\mathrm{d}x'=0. \qquad (4.1.17)
\end{aligned}$$

We combine (4.1.15), (4.1.16) and (4.1.17) to see that

$$\begin{aligned}
0&\le\lim_{\varepsilon\to 0^+}\int_{B_r^\varepsilon(x_0)}\mathrm{div}(|x_{m+1}|^{-\beta}\nabla|(D^{\tilde{\alpha}}v)^*|^2)\mathrm{d}x\\
&=\int_{\partial B_r(x_0)}|x_{m+1}|^{-\beta}\nu\cdot\nabla|(D^{\tilde{\alpha}}v)^*|^2\mathrm{d}\sigma(x).
\end{aligned}$$

This holds for every $r<R$. We apply Lemma 4.1.0.7, noting that

$$\int_{B_r(x_0)}|x_{m+1}|^{-\beta}|(D^{\tilde{\alpha}}v)^*|^2\mathrm{d}x=\int_{B_r(x_0)}|x_{m+1}|^{\beta}|\partial_{m+1}D^{\tilde{\alpha}}v|^2\mathrm{d}x,$$

to conclude the proof. □

Remark 4.1.0.2. This lemma indicates the derivatives $(D^{\tilde{\alpha}}v)^*$ corresponding to solutions of the Neumann problem (2.4.5) satisfy a stronger monotonicity property than stated in Theorem 4.1.0.1 if $\beta \in (-1,0)$ and a weaker such property if $\beta \in (0,1)$. We will make use of this fact to prove the theorem for $\beta \in (-1,0)$ in the following lemma.

Lemma 4.1.0.10. *Let $B_R(x_0) \subset \mathbb{R}^{m+1}$ with $(x_0)_{m+1} = 0$ and $R \leq 1$ and suppose $v \in W^{1,2}_\beta(B_R(x_0);\mathbb{R}^n)$ is a weak solution of div$(|x_{m+1}|^\beta \nabla v) = 0$ in $B_R(x_0)$. If v is symmetric with respect to $\partial\mathbb{R}^{m+1}_+$ in $B_R(x_0)$ and $\beta \in (-1,0)$ then*

$$s^{-(1+m+\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x \leq r^{-(1+m+\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x$$

for every $0 < s \leq r \leq R$.

Proof. Lemma 4.1.0.8 implies that

$$s^{-(1+m+\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta |\nabla' v|^2 \mathrm{d}x \leq r^{-(1+m+\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |\nabla' v|^2 \mathrm{d}x \quad (4.1.18)$$

for every $0 < s \leq r < R$. Moreover, Lemma 4.1.0.9 implies that

$$s^{-(1+m-\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta |\partial_{m+1} v|^2 \mathrm{d}x \leq r^{-(1+m-\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |\partial_{m+1} v|^2 \mathrm{d}x$$

for every $0 < s \leq r < R$. However, if $\beta < 0$ then $s^{-(1+m+\beta)} = s^{-2\beta} s^{-(1+m-\beta)}$ and hence

$$\begin{aligned}
& s^{-(1+m+\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta |\partial_{m+1} v|^2 \mathrm{d}x \\
&= s^{-2\beta} s^{-(1+m-\beta)} \int_{B_s(x_0)} |x_{m+1}|^\beta |\partial_{m+1} v|^2 \mathrm{d}x \\
&\leq s^{-2\beta} r^{-(1+m-\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |\partial_{m+1} v|^2 \mathrm{d}x \\
&= \left(\frac{s}{r}\right)^{-2\beta} r^{-(1+m+\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |\partial_{m+1} v|^2 \mathrm{d}x \\
&\leq r^{-(1+m+\beta)} \int_{B_r(x_0)} |x_{m+1}|^\beta |\partial_{m+1} v|^2 \mathrm{d}x. \qquad (4.1.19)
\end{aligned}$$

Adding together (4.1.18) and (4.1.19) concludes the proof of the lemma for all $0 < s \leq r < R$. As $v \in W^{1,2}_\beta(B_R(x_0);\mathbb{R}^n)$, we conclude the statement for $r = R$ using Lebesgue's Dominated Convergence Theorem. □

Remark 4.1.0.3. This lemma proves Theorem 4.1.0.1 when $\beta \in (-1,0)$.

It remains to show that Theorem 4.1.0.1 is satisfied when $\beta \in (0,1)$. Our method is similar to the proof of Lemma 4.1.0.9, but instead of considering the quantity $\operatorname{div}(|x_{m+1}|^{-\beta}\nabla|v^*|^2)$ we consider $\operatorname{div}(|x_{m+1}|^{\beta}\nabla|\partial_{m+1}v|^2)$.

Lemma 4.1.0.11. *Let $\beta \in (0,1)$ and suppose $v \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$ is a weak solution of $div(|x_{m+1}|^{\beta}\nabla v) = 0$ in $B_R(x_0)$ which is symmetric with respect to $\partial\mathbb{R}^{m+1}_{+}$. Let $\tilde{\alpha}$ be a multi-index with $\tilde{\alpha}_{m+1} = 0$. Then for every $0 < s \le r < R$ we have*

$$s^{-(1+m+\beta)}\int_{B_s(x_0)}|x_{m+1}|^{\beta}|\partial_{m+1}D^{\tilde{\alpha}}v|^2\mathrm{d}x \le r^{-(1+m+\beta)}\int_{B_r(x_0)}|x_{m+1}|^{\beta}|\partial_{m+1}D^{\tilde{\alpha}}v|^2\mathrm{d}x.$$

Proof. Recall that v is smooth in $B_R(x_0)\backslash\partial\mathbb{R}^{m+1}_{+}$. Fix $R > r > \varepsilon > 0$. Define $B^{\varepsilon}_r(x_0) = B_r(x_0) \cap \{x \in \mathbb{R}^{m+1} : |x_{m+1}| > \varepsilon\}$. We calculate

$$\begin{aligned}\operatorname{div}(|x_{m+1}|^{\beta}\nabla|\partial_{m+1}D^{\tilde{\alpha}}v|^2) = {}& 2|x_{m+1}|^{\beta}|\nabla\partial_{m+1}D^{\tilde{\alpha}}v|^2\\ &+ 2\langle\partial_{m+1}D^{\tilde{\alpha}}v, \operatorname{div}(|x_{m+1}|^{\beta}\nabla\partial_{m+1}D^{\tilde{\alpha}}v)\rangle\end{aligned} \tag{4.1.20}$$

classically in $B^{\varepsilon}_r(x_0)$. In contrast to the considerations of analogous terms in the proof of Lemmata 4.1.0.8 and 4.1.0.9, the term $\operatorname{div}(|x_{m+1}|^{\beta}\nabla\partial_{m+1}D^{\tilde{\alpha}}v)$ need not vanish. We can, however, still show that the right hand side above is positive when $\beta \in (0,1)$. As $\operatorname{div}(|x_{m+1}|^{\beta}\nabla D^{\tilde{\alpha}}v) = 0$ in $B^{\varepsilon}_r(x_0)$ we find

$$\begin{aligned}0 = \partial_{m+1}\operatorname{div}(|x_{m+1}|^{\beta}\nabla D^{\tilde{\alpha}}v) &= |x_{m+1}|^{\beta}\Delta\partial_{m+1}D^{\tilde{\alpha}}v\\ &\quad + \operatorname{sgn}(x_{m+1})\beta|x_{m+1}|^{\beta-1}\Delta D^{\tilde{\alpha}}v\\ &\quad + \operatorname{sgn}(x_{m+1})\beta|x_{m+1}|^{\beta-1}\partial^2_{m+1}D^{\tilde{\alpha}}v\\ &\quad + \beta(\beta-1)|x_{m+1}|^{\beta-2}\partial_{m+1}D^{\tilde{\alpha}}v\\ &= \operatorname{div}(|x_{m+1}|^{\beta}\nabla\partial_{m+1}D^{\tilde{\alpha}}v)\\ &\quad + \operatorname{sgn}(x_{m+1})\frac{\beta}{|x_{m+1}|}\operatorname{div}(|x_{m+1}|^{\beta}\nabla D^{\tilde{\alpha}}v)\\ &\quad - \beta|x_{m+1}|^{\beta-2}\partial_{m+1}D^{\tilde{\alpha}}v\\ &= \operatorname{div}(|x_{m+1}|^{\beta}\nabla\partial_{m+1}D^{\tilde{\alpha}}v)\\ &\quad - \beta|x_{m+1}|^{\beta-2}\partial_{m+1}D^{\tilde{\alpha}}v.\end{aligned}$$

Hence

$$\operatorname{div}(|x_{m+1}|^{\beta}\nabla\partial_{m+1}D^{\tilde{\alpha}}v) = \beta|x_{m+1}|^{\beta-2}\partial_{m+1}D^{\tilde{\alpha}}v \tag{4.1.21}$$

in $B_r^\varepsilon(x_0)$. We conclude from (4.1.20) and (4.1.21) that

$$\begin{aligned}\operatorname{div}(|x_{m+1}|^\beta\nabla|\partial_{m+1}D^{\tilde\alpha}v|^2) = 2|x_{m+1}|^\beta|\nabla\partial_{m+1}D^{\tilde\alpha}v|^2 \\ + 2\beta|x_{m+1}|^{\beta-2}|\partial_{m+1}D^{\tilde\alpha}v|^2 \geq 0.\end{aligned}$$

An application of the Divergence Theorem on $B_r^\varepsilon(x_0)$ yields

$$\begin{aligned}0 &\leq \int_{B_r^\varepsilon(x_0)} \operatorname{div}(|x_{m+1}|^\beta\nabla|\partial_{m+1}D^{\tilde\alpha}v|^2)\mathrm{d}x \\ &= \int_{\partial B_r(x_0)} \mathbb{1}_{\overline{B_r^\varepsilon(x_0)}}|x_{m+1}|^\beta\nu\cdot\nabla|\partial_{m+1}D^{\tilde\alpha}v|^2\mathrm{d}\sigma(x) \\ &\quad - \int_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \varepsilon^\beta e_{m+1}\cdot\nabla(|\partial_{m+1}D^{\tilde\alpha}v|^2(x',\varepsilon) - |\partial_{m+1}D^{\tilde\alpha}v|^2(x',-\varepsilon))\mathrm{d}x'\end{aligned} \tag{4.1.22}$$

where $\mathbb{1}_{\overline{B_r^\varepsilon(x_0)}}$ is the indicator function of $\overline{B_r^\varepsilon(x_0)}$ and ν is the outward unit normal on $\partial B_r(x_0)$. We consider the terms on the right hand side above separately with a view to taking the limit as $\varepsilon \to 0^+$.

We recall the required regularity results for v. Observe that each $D^{\tilde\alpha}v$ is continuous in $B_R(x_0)$ and symmetric with respect to $\partial\mathbb{R}^{m+1}_+$ by Lemma 4.1.0.4. For every $r < R$, each $D^{\tilde\alpha}v \in W^{1,2}_\beta(B_r(x_0);\mathbb{R}^n)$ and is a weak solution of $\operatorname{div}(|x_{m+1}|^\beta\nabla D^{\tilde\alpha}v) = 0$ in $B_r(x_0)$ by Lemma 4.1.0.3. Lemma 4.1.0.6 implies that $(D^{\tilde\alpha}v)^* = |x_{m+1}|^\beta\partial_{m+1}D^{\tilde\alpha}v$ is continuous in $B_R(x_0)$ with $(D^{\tilde\alpha}v)^*|_{\partial^0B_R^+(x_0)} = 0$. Hence $(D^{\tilde\alpha}v)^*$ is bounded in $\overline{B_r(x_0)}$ for every $r < R$. We also note that $|x_{m+1}|^{-\beta}\partial_{m+1}(D^{\tilde\alpha}v)^* = -\Delta'D^{\tilde\alpha}v$ is continuous in $B_R(x_0)$ and thus also bounded in $\overline{B_r(x_0)}$ for every $r < R$. The same is true for $\partial_{m+1}(D^{\tilde\alpha}v)^* = -|x_{m+1}|^\beta\Delta'D^{\tilde\alpha}v$ and we see additionally that $\partial_{m+1}(D^{\tilde\alpha}v)^*|_{\partial^0B_R^+(x_0)} = 0$. An application of Lemma 4.1.0.4 implies that for every $r < R$ and $i = 1,\ldots,m$, the derivatives $\partial_i(D^{\tilde\alpha}v)^*$ are continuous in $B_r(x_0)$. The preceding statements hold for every multi-index $\tilde\alpha$ with $\tilde\alpha_{m+1} = 0$.

Now we show that the derivative $\partial_{m+1}D^{\tilde\alpha}v(x',0)$ exists and is equal to 0 for $(x',0) \in \partial^0B_R^+(x_0)$. Fix such an $(x',0)$, note that $(x',0) \in B_r(x_0)$ for some $r < R$ and choose h with $|h|$ sufficiently small as to ensure $(x',h) \in B_r(x_0)$. Using the aforementioned properties of v, together with the Mean Value Theorem, we find

$$\begin{aligned} |h|^{-1}|D^{\tilde{\alpha}}v(x',h) - D^{\tilde{\alpha}}v(x',0)| &= |\partial_{m+1}D^{\tilde{\alpha}}v(x',x_{m+1})| \\ &= |x_{m+1}|^{-\beta}|(D^{\tilde{\alpha}}v)^*(x',x_{m+1}) - 0| \\ &= |x_{m+1}|^{-\beta}|\partial_{m+1}(D^{\tilde{\alpha}}v)^*(x',\xi)||x_{m+1}| \\ &\leq |x_{m+1}|||\xi|^{-\beta}\partial_{m+1}(D^{\tilde{\alpha}}v)^*(x',\xi)| \\ &\leq C|h| \to 0 \text{ as } h \to 0, \end{aligned} \tag{4.1.23}$$

where x_{m+1} with $|x_{m+1}| \in (0,|h|)$ and ξ with $|\xi| \in (0,|x_{m+1}|)$ are chosen such that the Mean Value Theorem holds. Notice that (4.1.23) holds for every $(x',0) \in \partial^0 B_R^+(x_0)$ and thus we see that $\partial_{m+1}D^{\tilde{\alpha}}v$ exists and is equal to zero on $\partial^0 B_R^+(x_0)$. Similar calculations to those on the right hand side of (4.1.23) show that $\partial_{m+1}D^{\tilde{\alpha}}v$ is continuous at every point in $\partial^0 B_R^+(x_0)$. Noting that $\partial_{m+1}D^{\tilde{\alpha}}v$ is smooth in $B_R(x_0)\backslash\partial\mathbb{R}_+^{m+1}$, we conclude it is continuous in $B_R(x_0)$.

Next we show $|x_{m+1}|^\beta\partial_{m+1}^2D^{\tilde{\alpha}}v$ is bounded in $\overline{B_r(x_0)}\backslash\partial\mathbb{R}_+^{m+1}$ for every $r < R$; we already know it is continuous in $B_R(x_0)\backslash\partial\mathbb{R}_+^{m+1}$. We calculate

$$|x_{m+1}|^\beta\partial_{m+1}^2D^{\tilde{\alpha}}v = \partial_{m+1}(D^{\tilde{\alpha}}v)^* - \beta x_{m+1}^{-1}(D^{\tilde{\alpha}}v)^*. \tag{4.1.24}$$

Recall that $(D^{\tilde{\alpha}}v)^* = 0$ on $\partial^0 B_R^+(x_0)$ and $(D^{\tilde{\alpha}}v)^*, \partial_{m+1}(D^{\tilde{\alpha}}v)^*$ are continuous in $B_R(x_0)$ and hence bounded in $\overline{B_r(x_0)}$ for every $r < R$. We therefore deduce, using the Mean Value Theorem, that for every $r < R$ and every point in $\overline{B_r(x_0)}\backslash\partial\mathbb{R}_+^{m+1}$ we have

$$\begin{aligned} ||x_{m+1}|^\beta\partial_{m+1}^2D^{\tilde{\alpha}}v| &\leq |\partial_{m+1}(D^{\tilde{\alpha}}v)^*| + \beta|x_{m+1}^{-1}(D^{\tilde{\alpha}}v)^*| \\ &= |\partial_{m+1}(D^{\tilde{\alpha}}v)^*| + \beta|\partial_{m+1}(D^{\tilde{\alpha}}v)^*(x',\xi)| \\ &\leq C, \end{aligned}$$

where ξ is chosen with $|\xi| \in (0,|x_{m+1}|)$ such that the Mean Value Theorem holds.

We now return to (4.1.22). We have shown above that $D^{\tilde{\alpha}}\partial_{m+1}v$ is continuous in $B_R(x_0)$ and $|x_{m+1}|^\beta\nabla D^{\tilde{\alpha}}\partial_{m+1}v$ is bounded in $\overline{B_r(x_0)}\backslash\partial\mathbb{R}_+^{m+1}$ for every $r < R$. It follows that $|x_{m+1}|^\beta\nu\cdot\nabla|D^{\tilde{\alpha}}\partial_{m+1}v|^2$ is bounded on $\overline{B_r(x_0)}\backslash\partial\mathbb{R}_+^{m+1}$ for every $r < R$. Applying Lebesgue's Dominated Convergence Theorem, we see that

$$\begin{aligned} &\lim_{\varepsilon\to 0^+}\int_{\partial B_r(x_0)} \mathbb{1}_{\overline{B_r^\varepsilon(x_0)}}|x_{m+1}|^\beta\nu\cdot\nabla|D^{\tilde{\alpha}}\partial_{m+1}v|^2\mathrm{d}\sigma(x) \\ &= \int_{\partial B_r(x_0)}|x_{m+1}|^\beta\nu\cdot\nabla|D^{\tilde{\alpha}}\partial_{m+1}v|^2\mathrm{d}\sigma(x). \end{aligned} \tag{4.1.25}$$

Now we deal with the other term on the right hand side of (4.1.22). We observe that $\partial^2_{m+1} D^{\tilde{\alpha}} v(x',\varepsilon) = \partial^2_{m+1} D^{\tilde{\alpha}} v(x',-\varepsilon)$ and $\partial_{m+1} D^{\tilde{\alpha}} v(x',\varepsilon) = -\partial_{m+1} D^{\tilde{\alpha}} v(x',-\varepsilon)$. Thus

$$\begin{aligned}
& e_{m+1} \cdot \nabla(|\partial_{m+1} D^{\tilde{\alpha}} v|^2(x',\varepsilon) - |\partial_{m+1} D^{\tilde{\alpha}} v|^2(x',-\varepsilon)) \\
&= 2\left\langle \partial^2_{m+1} D^{\tilde{\alpha}} v(x',\varepsilon), \partial_{m+1} D^{\tilde{\alpha}} v(x',\varepsilon)\right\rangle - 2\left\langle \partial^2_{m+1} D^{\tilde{\alpha}} v(x',-\varepsilon), \partial_{m+1} D^{\tilde{\alpha}} v(x',-\varepsilon)\right\rangle \\
&= 4\left\langle \partial^2_{m+1} D^{\tilde{\alpha}} v(x',\varepsilon), \partial_{m+1} D^{\tilde{\alpha}} v(x',\varepsilon)\right\rangle .
\end{aligned}$$

Notice that $\mathbb{1}_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \partial_{m+1} D^{\tilde{\alpha}} v(x',\varepsilon)$ converges uniformly, for $x' \in \overline{B^m_r(x_0)}$, to 0 as $\varepsilon \to 0^+$. Furthermore, $D^{\tilde{\alpha}} \partial_{m+1} v$ and $|x_{m+1}|^\beta \partial^2_{m+1} D^{\tilde{\alpha}} \partial_{m+1} v$ are uniformly bounded in $\overline{B_r(x_0)}$. Thus $\mathbb{1}_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \varepsilon^\beta \left\langle \partial^2_{m+1} D^{\tilde{\alpha}} v(x',\varepsilon), \partial_{m+1} D^{\tilde{\alpha}} v(x',\varepsilon)\right\rangle \to 0$ uniformly as $\varepsilon \to 0^+$. We may therefore infer that

$$\begin{aligned}
& \lim_{\varepsilon \to 0^+} \int_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \varepsilon^\beta e_{m+1} \cdot \nabla(|\partial_{m+1} D^{\tilde{\alpha}} v|^2(x',\varepsilon) - |\partial_{m+1} D^{\tilde{\alpha}} v|^2(x',-\varepsilon)) \mathrm{d}x' \\
&= 4 \lim_{\varepsilon \to 0^+} \int_{B^m_{\sqrt{r^2-\varepsilon^2}}(x_0)} \varepsilon^\beta \left\langle \partial^2_{m+1} D^{\tilde{\alpha}} v(x',\varepsilon), \partial_{m+1} D^{\tilde{\alpha}} v(x',\varepsilon)\right\rangle \mathrm{d}x' = 0. \qquad (4.1.26)
\end{aligned}$$

We combine (4.1.22), (4.1.25) and (4.1.26) to see that

$$\begin{aligned}
0 &\le \lim_{\varepsilon \to 0^+} \int_{B^\varepsilon_r(x_0)} \operatorname{div}(|x_{m+1}|^\beta \nabla |\partial_{m+1} D^{\tilde{\alpha}} v|^2) \mathrm{d}x \\
&= \int_{\partial B_r(x_0)} |x_{m+1}|^\beta \nu \cdot \nabla |\partial_{m+1} D^{\tilde{\alpha}} v|^2 \mathrm{d}\sigma(x). \qquad (4.1.27)
\end{aligned}$$

This holds on every $B_r(x_0)$ with $r < R$ and we apply Lemma 4.1.0.7 to conclude the proof. □

Proof of Theorem 4.1.0.1. The conclusion of Lemma 4.1.0.8 combined with the conclusion of Lemma 4.1.0.11 prove Theorem 4.1.0.1 when $\beta \in (0,1)$. Since the theorem is also proved for $\beta \in (-1,0)$ in Lemma 4.1.0.10 and is well known for $\beta = 0$, we conclude the theorem holds for every $\beta \in (-1,1)$. □

4.2 Solutions of the Linear Degenerate Dirichlet Problem

Here we discuss solutions of the Dirichlet problem

$$\operatorname{div}(|x_{m+1}|^\beta \nabla v) = 0 \text{ in } B_R(x_0) \qquad \text{and} \qquad v = \phi \text{ on } \partial B_R(x_0) \qquad (4.2.1)$$

for a given ϕ. This type of problem has been considered extensively in the literature. An exposition of some of the theory is given in [25] and we record the results we need in the following lemma.

Lemma 4.2.0.1. *Suppose $\phi \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$. Then there exists a $v \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$ which is a weak solution of the Dirichlet problem* (4.2.1). *In other words, v is a weak solution of div$(|x_{m+1}|^{\beta}\nabla v) = 0$ in $B_R(x_0)$ with $v - \phi \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$. Moreover, the following holds. Any weak solution is unique and continuous in $B_R(x_0)$, if $\phi \in C(\overline{B_R(x_0)};\mathbb{R}^n)$ then $v(x) \to \phi(z)$ as $x \to z$ for $z \in \partial B_R(x_0)$ and the weak maximum principle*

$$\max_{\overline{B_R(x_0)}} v = \max_{\partial B_R(x_0)} v = \max_{\partial B_R(x_0)} \phi$$

and weak minimum principle

$$\min_{\overline{B_R(x_0)}} v = \min_{\partial B_R(x_0)} v = \min_{\partial B_R(x_0)} \phi$$

both hold, where we take the maximum and minimum component-wise. For any $\phi \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$, if v is a weak solution of the Dirichlet problem (4.2.1) *and $w \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$ also satisfies $w - \phi \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$ then*

$$\int_{B_R(x_0)} |x_{m+1}|^{\beta}|\nabla v|^2 \mathrm{d}x \leq \int_{B_R(x_0)} |x_{m+1}|^{\beta}|\nabla w|^2 \mathrm{d}x.$$

Proof. Recall that $|x_{m+1}|^{\beta}$ is of Muckenhoupt class A_2. Thus according to 1.6 of [25], $|x_{m+1}|^{\beta}$ is a 2-admissible weight which, in particular, means we may apply the results there. For complete details see 1.1 of [25]. Theorem 3.70 in [25] asserts that any weak solution of div$(|x_{m+1}|^{\beta}\nabla v) = 0$ in $B_R(x_0)$ is continuous. Moreover, if $v \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$ is a weak solution of div$(|x_{m+1}|^{\beta}\nabla v) = 0$ in $B_R(x_0)$, $\phi \in C(\overline{B_R(x_0)};\mathbb{R}^n) \cap W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$ and $v - \phi \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$ it follows from corollary 6.32 in [25] that $v(x) \to \phi(z)$ as $x \to z$ for $z \in \partial B_R(x_0)$. Furthermore, the strong maximum principle, 6.5 in [25], then immediately implies the weak maximum principle and weak minimum principle as stated in the lemma.

The Dirichlet problem is uniquely solvable according to 3.17 in [25]. Lastly we show that solutions of (4.2.1) are energy minimising. Consider a weak solution $v \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$ of div$(|x_{m+1}|^{\beta}\nabla v) = 0$ in $B_R(x_0)$ with $v - \phi \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$ and let $w \in W^{1,2}_{\beta}(B_R(x_0);\mathbb{R}^n)$ with $w - \phi \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$.

It follows that $w - v \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$. We calculate

$$\begin{aligned}\int_{B_R(x_0)} |x_{m+1}|^\beta |\nabla w|^2 \mathrm{d}x &= \int_{B_R(x_0)} |x_{m+1}|^\beta |\nabla v + \nabla(w-v)|^2 \mathrm{d}x \\ &= \int_{B_R(x_0)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x + \int_{B_R(x_0)} |x_{m+1}|^\beta |\nabla (w-v)|^2 \mathrm{d}x \\ &\quad + 2\int_{B_R(x_0)} |x_{m+1}|^\beta \langle \nabla v, \nabla(w-v)\rangle \mathrm{d}x.\end{aligned}$$

However, since v is a weak solution of $\operatorname{div}(|x_{m+1}|^\beta \nabla v) = 0$ in $B_R(x_0)$ and $w - v \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$, it follows from Remark 2.4.0.1 that

$$\int_{B_R(x_0)} |x_{m+1}|^\beta \langle \nabla v, \nabla(w-v)\rangle \mathrm{d}x = 0.$$

Hence

$$\int_{B_R(x_0)} |x_{m+1}|^\beta |\nabla w|^2 \mathrm{d}x = \int_{B_R(x_0)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x + \int_{B_R(x_0)} |x_{m+1}|^\beta |\nabla (w-v)|^2 \mathrm{d}x$$

which concludes the proof. □

We now consider solutions to the Dirichlet problem (4.2.1) with boundary data which is symmetric with respect to $\partial\mathbb{R}^{m+1}_+$.

Lemma 4.2.0.2. *Suppose $v, \phi \in W^{1,2}_\beta(B_R(x_0);\mathbb{R}^n)$ and v is a weak solution of the Dirichlet problem* (4.2.1) *with ϕ as boundary data. Let $\phi \in C(\overline{B_R(x_0)};\mathbb{R}^n)$ and suppose $\phi(x', x_{m+1}) = \phi(x', -x_{m+1})$ for every $(x', x_{m+1}) \in \overline{B_R(x_0)}$. Then $v(x', x_{m+1}) = v(x', -x_{m+1})$ for every $(x', x_{m+1}) \in B_R(x_0)$.*

Proof. Our goal is to show that $\tilde{v}(x', x_{m+1}) := v(x', -x_{m+1})$ is a continuous weak solution of the same Dirichlet problem (4.2.1) as v. Since solutions to this problem are unique as a consequence of Lemma 4.2.0.1, we then have $\tilde{v} = v$, which implies v is symmetric with respect to $\partial\mathbb{R}^{m+1}_+$ as required.

Since ϕ is continuous in $\overline{B_R(x_0)}$, Lemma 4.2.0.1 implies that v is continuous in $\overline{B_R(x_0)}$. Hence, so is the function $\tilde{v}(x', x_{m+1}) := v(x', -x_{m+1})$. Furthermore, $\tilde{v}|_{\partial B_R(x_0)} = \phi|_{\partial B_R(x_0)}$ since $\phi(x', x_{m+1}) = \phi(x', -x_{m+1})$ for every $(x, x_{m+1}) \in \overline{B_R(x_0)}$. We also note $\tilde{v} \in W^{1,2}_\beta(B_R(x_0);\mathbb{R}^n)$. Hence $\tilde{v} - \phi \in W^{1,2}_{\beta,0}(B_R(x_0);\mathbb{R}^n)$. Finally we will show that $\tilde{v}$ weakly satisfies $\operatorname{div}(|x_{m+1}|^\beta \nabla \tilde{v}) = 0$ in $B_R(x_0)$. Let

$\psi \in C_0^\infty(B_R(x_0);\mathbb{R}^n)$. We have

$$\int_{B_R(x_0)} |x_{m+1}|^\beta \langle \nabla\tilde{v}, \nabla\psi\rangle \,\mathrm{d}x = \int_{B_R(x_0)} |x_{m+1}|^\beta \langle \nabla'\tilde{v}, \nabla'\psi\rangle \,\mathrm{d}x + \int_{B_R(x_0)} |x_{m+1}|^\beta \langle \partial_{m+1}\tilde{v}, \partial_{m+1}\psi\rangle \,\mathrm{d}x. \tag{4.2.2}$$

We consider the right hand side of (4.2.2). Define $\tilde{\psi}(x', x_{m+1}) = \tilde{\psi}(x', -x_{m+1})$ and observe that $\tilde{\psi} \in C_0^\infty(B_R(x_0);\mathbb{R}^n)$. Using the change of variables $x_{m+1} \mapsto -x_{m+1}$ we calculate

$$\int_{B_R(x_0)} |x_{m+1}|^\beta \langle \nabla'\tilde{v}, \nabla'\psi\rangle \,\mathrm{d}x = \int_{B_R(x_0)} |x_{m+1}|^\beta \left\langle \nabla' v, \nabla'\tilde{\psi}\right\rangle \mathrm{d}x \tag{4.2.3}$$

and

$$\int_{B_R(x_0)} |x_{m+1}|^\beta \langle \partial_{m+1}\tilde{v}, \partial_{m+1}\psi\rangle \,\mathrm{d}x = \int_{B_R(x_0)} |x_{m+1}|^\beta \left\langle \partial_{m+1} v, \partial_{m+1}\tilde{\psi}\right\rangle \mathrm{d}x. \tag{4.2.4}$$

Together, (4.2.2), (4.2.3) and (4.2.4) imply

$$\int_{B_R(x_0)} |x_{m+1}|^\beta \langle \nabla\tilde{v}, \nabla\psi\rangle \,\mathrm{d}x = \int_{B_R(x_0)} |x_{m+1}|^\beta \left\langle \nabla v, \nabla\tilde{\psi}\right\rangle \mathrm{d}x = 0. \tag{4.2.5}$$

Hence v and $\tilde{v}$ solve the same Dirichlet problem and therefore Lemma 4.2.0.1 implies $\tilde{v} = v$ which concludes the proof. □

Chapter 5

Hölder Continuity of First Derivatives of Minimisers

In Chapter 3 we established the Hölder continuity of a minimiser v of E^β relative to $\mathcal{O}$. In doing so, we took advantage of the fact that v is a harmonic map in $\mathbb{R}^{m+1}_+$ and the restriction of the metric g, defined by (3.0.1) in Chapter 3, is smooth and bounded on compact sets in $\mathbb{R}^{m+1}_+$; in this case if v is continuous, then it is smooth. The higher regularity of continuous harmonic maps was known when the first partial regularity theories for harmonic maps were developed and there are several ways to prove that continuity implies higher regularity. We proceed to highlight the theory upon which the methods in this chapter are based.

Jost, see Chapter 8 of [27], gives an exposition of one possible way to prove continuous harmonic maps are smooth. The approach is based on the theory, developed by Ladyzhenskaya, for linear and quasilinear uniformly elliptic equations [28]. The application of difference quotients to establish the existence and integrability of higher order derivatives is one of the main constituents of the method. This technique is also widely used throughout the literature on elliptic equations and, in order to adapt the technique to the theory of harmonic maps, the continuity assumption must be exploited due to the nature of the Euler-Lagrange equations these maps satisfy.

If a harmonic map is Hölder continuous, it is possible to bypass some of the technicalities in the approach described by Jost. The average energy of a harmonic map is comparable to that of a solution to the associated linear equation with the same boundary data. In conjunction with the scaling and minimising properties of such a solution, the Hölder continuity of a harmonic map can be used to derive a bound for the essential supremum of its gradient as described by Schoen [45].

Once sufficient integrability of the higher order derivatives, or a bound for the supremum of the gradient, of a harmonic map is known, higher regularity follows from repeated differentiation of the Euler-Lagrange equations coupled with applications of the Sobolev embedding theorem. If we view the Euler-Lagrange equations for E^β, recall (3.1.10), as elliptic equations, we see that the ellipticity degenerates at the boundary $\partial\mathbb{R}^{m+1}_+$. Furthermore, the derivatives of the coefficient x^β_{m+1} become increasingly singular if we formally repeatedly differentiate the Euler-Lagrange equations for E^β with respect to x_{m+1}. Therefore, we do not necessary expect to be able to establish higher regularity of all higher order derivatives of a minimiser of E^β up to $\partial\mathbb{R}^{m+1}_+$ using the Sobolev embedding theorem and methods from the regularity theory for harmonic maps directly.

Ultimately, as will be discussed in Chapter 6, we are interested in a variational problem for maps defined on $\mathcal{O} \subset \partial\mathbb{R}^{m+1}_+$. We therefore focus on establishing higher regularity of a minimiser of E^β relative to $\mathcal{O}$ in the directions x_i, for $i = 1, \dots, m$, up to the boundary. We observed in Chapter 4 that for solutions of the linear equation $\operatorname{div}(|x_{m+1}|^\beta \nabla v) = 0$, the regularity for the derivatives with respect to x_i, for $i = 1, \dots, m$, can be established without considering the derivatives with respect to x_{m+1}. In this chapter we show that the same is essentially true for the derivatives $\partial_i v$, where $i = 1, \dots, m$.

We have a substitute for the general Sobolev embedding theorem on balls with centre in $\partial\mathbb{R}^{m+1}_+$, namely the modified lemma of Morrey, Lemma 3.5.0.2, which we used to establish the continuity of minimisers v of E^β relative to $\mathcal{O}$. The goal of the method presented here is to again show decay estimates, this time for the scaled energy of derivatives of v with respect to x_i for $i = 1, \dots, m$, which allow us do deduce (3.5.3) and (3.5.4) in Lemma 3.5.0.2. We observe that the aforementioned derivatives of a minimiser of E^β relative to $\mathcal{O}$ satisfy a second order equation with better structural conditions than those of the Euler-Lagrange equations for E^β. Consequently, we prove a Cacciopoli-type inequality which is instrumental in the proof of the decay estimates.

Recall the assumptions on m, β stated in Remark 2.2.1.1. We make the same assumptions throughout this chapter.

5.1 An L^∞ Bound for the Gradient of an Energy Minimiser

We show that the combination of Hölder continuity and sufficiently small energy of minimiser of E^β relative to $\mathcal{O}$ implies a bound for the essential supremum of its gradient. The following lemma is an analogue of Lemma 3.1 in [45] and the proof given here is based on the proof in [45].

Lemma 5.1.0.1. *Suppose* $v \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_{+}; N)$ *is a minimiser of* E^{β} *relative to* $\mathcal{O}$. *Let* $\varepsilon > 0$ *be the number from Theorem 3.12.1.1 and let* $B^{+}_{R}(x_0)$ *be a half-ball with* $R \leq 1$ *and* $\overline{\partial^0 B^{+}_{R}(x_0)} \subset \mathcal{O}$. *Suppose* $R^{1-m-\beta}\int_{B^{+}_{R}(x_0)} x^{\beta}_{m+1}|\nabla v|^2 \mathrm{d}x \leq \varepsilon$. *Then there is a* $\theta = \theta(m, N, \beta) \in (0,1)$ *such that* $\nabla v \in L^{\infty}(B^{+}_{\theta R}(x_0); \mathbb{R}^{(m+1)n})$. *In particular, we have*

$$||\nabla v||^2_{L^{\infty}(B^{+}_{\theta R}(x_0);\mathbb{R}^{(m+1)n})} \leq C\frac{1}{|B^{+}_{3\theta R}(x_0)|_{\beta}}\int_{B^{+}_{3\theta R}(x_0)} x^{\beta}_{m+1}|\nabla v|^2 \mathrm{d}x$$

where $C = C(m, N, \beta)$.

Proof. It follows from Theorem 3.12.1.1 that there is a $\theta_0 \in (0,1)$ such that v is Hölder continuous in $B^{+}_{\theta_0 R}(x_0)$. That is, there is a $\gamma = \gamma(m, N, \beta) \in (0,1)$ such that

$$|v(x) - v(y)| \leq \frac{C}{R^{\gamma}}|x-y|^{\gamma} \tag{5.1.1}$$

for every $x, y \in B^{+}_{\theta_0 R}(x_0)$. Recall from the proof of Theorem 3.12.1.1 and (3.6.24) in the proof of Lemma 3.6.0.4, that ε is also chosen sufficiently small to imply Lemma 3.6.0.3 holds, that is, we have

$$||\nabla v||^2_{L^{\infty}(B_{\frac{\rho}{2}}(y);\mathbb{R}^{(m+1)n})} \leq C\frac{1}{|B_{\rho}(y)|}\int_{B_{\rho}(y)} |\nabla v|^2 \mathrm{d}x, \tag{5.1.2}$$

on every $B_{\rho}(y) \in \mathcal{B}_{\theta_1}(x_0, R, \frac{R}{3})$ where $\theta_1 = \theta_1(m, N) \geq 2$. We intend to show a uniform bound for $||\nabla v||^2_{L^{\infty}(B_{\frac{y_{m+1}}{2\theta_1}}(y);\mathbb{R}^{(m+1)n})}$ where $B_{\frac{y_{m+1}}{\theta_1}}(y) \in \mathcal{B}_{\theta_1}(x_0, \theta_0 R, \frac{\theta_0 R}{3})$; on this class of ball we have

$$B_{\frac{y_{m+1}}{\theta_1}}(y) \subset B^{+}_{\frac{\theta_1+1}{\theta_1}y_{m+1}}(y^{+}) \subset B^{+}_{\frac{\theta_0 R}{2}}(y^{+}) \subset B^{+}_{\theta_0 R}(x_0) \tag{5.1.3}$$

so we may take advantage of the Hölder continuity of v on $B^{+}_{\theta_0 R}(x_0)$.

To obtain the required bound, in view of (5.1.3) we examine the decay of the average energy on concentric balls $B_{\rho}(y)$ with $B^{+}_{\rho}(y) \in \mathcal{B}^{+}(x_0, \theta_0 R, \frac{\theta_0 R}{2})$. Without relabelling, we reflect v evenly across the hyperplane $\mathbb{R}^m \times \{0\}$. Then $v \in C^{0,\gamma}(\overline{B_{\theta_0 R}(x_0)}; N) \cap W^{1,2}_{\beta}(B_R(x_0); N)$ is a weak solution of $\mathrm{div}(|x_{m+1}|^{\beta}\nabla v) + |x_{m+1}|^{\beta}A(v)(\nabla v, \nabla v) = 0$ in $B_R(x_0)$. Let $B^{+}_{\rho}(y) \in \mathcal{B}^{+}(x_0, \theta_0 R, \frac{\theta_0 R}{2})$. We focus initially on an estimate for the average energy on $B_{\frac{\rho}{2}}(y)$ in terms of that on $B_{\rho}(y)$.

An application of Minkowski's inequality, for maps in $L^2_\beta(B_{\frac{\varrho}{2}}(y);\mathbb{R}^{n(m+1)})$, yields

$$\left(\int_{B_{\frac{\varrho}{2}}(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}} \leq \left(\int_{B_{\frac{\varrho}{2}}(y)} |x_{m+1}|^\beta \left|\nabla w\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_{B_\rho(y)} |x_{m+1}|^\beta \left\langle \nabla(v-w), \nabla(v-w)\right\rangle \mathrm{d}x\right)^{\frac{1}{2}} \tag{5.1.4}$$

for any $w \in W^{1,2}_\beta(B_\rho(y);\mathbb{R}^n)$. Now suppose $w \in W^{1,2}_\beta(B_\rho(y);\mathbb{R}^n)$ is the weak solution of $\operatorname{div}(|x_{m+1}|^\beta \nabla w) = 0$ in $B_\rho(y)$ with $w = v$ on $\partial B_\rho(y)$, given by Lemma 4.2.0.1. Then w is smooth in $B_\rho(y)\backslash(\mathbb{R}^m \times \{0\})$ and continuous in $\overline{B_\rho(y)}$ by Lemma 4.2.0.1. Furthermore, since v is symmetric with respect to $\partial\mathbb{R}^{m+1}_+$, it follows from Lemma 4.2.0.2 that w is symmetric with respect to $\partial\mathbb{R}^{m+1}_+$ and, crucially, we are now free to apply Theorem 4.1.0.1 on any $B_\rho(y)$ with $B^+_\rho(y) \in \mathcal{B}^+(x_0, \theta_0 R, \frac{\theta_0 R}{2})$.

Recall $w - v \in C(\overline{B_\rho(y)};\mathbb{R}^n) \cap W^{1,2}_{\beta,0}(B_\rho(y);\mathbb{R}^n)$, v satisfies $\operatorname{div}(|x_{m+1}|^\beta \nabla v) + |x_{m+1}|^\beta A(v)(\nabla v, \nabla v) = 0$ weakly in $B_\rho(y)$ and w weakly solves $\operatorname{div}(|x_{m+1}|^\beta \nabla w) = 0$ in $B_\rho(y)$. We may therefore expand the second term in the right hand side of (5.1.4) to see that

$$\begin{aligned}
&\int_{B_\rho(y)} |x_{m+1}|^\beta \left\langle \nabla(v-w), \nabla(v-w)\right\rangle \mathrm{d}x \\
&= \int_{B_\rho(y)} |x_{m+1}|^\beta \left\langle \nabla v, \nabla(v-w)\right\rangle \mathrm{d}x \\
&\quad - \int_{B_\rho(y)} |x_{m+1}|^\beta \left\langle \nabla w, \nabla(v-w)\right\rangle \mathrm{d}x \\
&= \int_{B_\rho(y)} |x_{m+1}|^\beta \left\langle \nabla v, \nabla(v-w)\right\rangle \mathrm{d}x \\
&= \int_{B_\rho(y)} |x_{m+1}|^\beta \left\langle v-w, A(v)(\nabla v, \nabla v)\right\rangle \mathrm{d}x \\
&\leq C \sup_{B_\rho(y)} |v-w| \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x.
\end{aligned}$$

Hence, substituting this bound into (5.1.4), we have

$$\left(\int_{B_{\frac{\varrho}{2}}(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}} \leq \left(\int_{B_{\frac{\varrho}{2}}(y)} |x_{m+1}|^\beta |\nabla w|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \left(C \sup_{B_\rho(y)} |v-w| \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Now we divide by $|B_{\frac{\varrho}{2}}(y)|_\beta^{\frac{1}{2}}$, this gives

$$\begin{aligned}
&\left(\frac{1}{|B_{\frac{\varrho}{2}}(y)|_\beta} \int_{B_{\frac{\varrho}{2}}(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{|B_{\frac{\varrho}{2}}(y)|_\beta} \int_{B_{\frac{\varrho}{2}}(y)} |x_{m+1}|^\beta |\nabla w|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\quad + \left(C \sup_{B_\rho(y)} |v-w| \frac{1}{|B_\rho(y)|_\beta} \int_{B_\rho(y)} |x_{m+1}|^\beta |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}}. \qquad (5.1.5)
\end{aligned}$$

We consider the terms on the right hand side above.

Fix $z \in \partial B_\rho(y)$. Then $v(z) = w(z)$ and it follows that

$$\begin{aligned}
\sup_{B_\rho(y)} |v(x) - w(x)| &\leq \sup_{B_\rho(y)} |v(x) - v(z)| + \sup_{B_\rho(y)} |v(z) - w(x)| \\
&= \sup_{B_\rho(y)} |v(x) - v(z)| + \sup_{B_\rho(y)} |w(z) - w(x)|.
\end{aligned}$$

The Hölder continuity of v, together with (5.1.1), yields

$$\sup_{B_\rho(y)} |v(x) - v(z)| \leq C \frac{\rho^\gamma}{R^\gamma}.$$

We also note that w satisfies the weak maximum and minimum principle componentwise by lemma 4.2.0.1. It follows that, for $k = 1, \ldots, n$, we have

$$\sup_{B_\rho(y)} |w^k(x) - w^k(z)| \leq \sup_{\partial B_\rho(y)} |w^k(x) - w^k(z)| = \sup_{\partial B_\rho(y)} |v^k(x) - v^k(z)| \leq C \frac{\rho^\gamma}{R^\gamma}.$$

We thus conclude that

$$\sup_{B_\rho(y)} |v(x) - w(x)| \leq C \frac{\rho^\gamma}{R^\gamma}. \qquad (5.1.6)$$

Next we use the monotonicity and minimising properties of w to scale its averaged energy in (5.1.5). It follows from an application of Theorem 4.1.0.1, followed by an application of Lemma 4.2.0.1, that

$$\begin{aligned}\frac{1}{|B_{\frac{\rho}{2}}(y)|_\beta}\int_{B_{\frac{\rho}{2}}(y)}|x_{m+1}|^\beta|\nabla w|^2\mathrm{d}x &\leq \frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla w|^2\mathrm{d}x\\ &\leq \frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x. \end{aligned}\tag{5.1.7}$$

Hence, combining (5.1.5), (5.1.6) and (5.1.7), we have

$$\begin{aligned}&\left(\frac{1}{|B_{\frac{\rho}{2}}(y)|_\beta}\int_{B_{\frac{\rho}{2}}(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\right)^{\frac{1}{2}}\\ &\leq \left(\frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\right)^{\frac{1}{2}}\\ &\quad+\left(C\frac{\rho^\gamma}{R^\gamma}\frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\right)^{\frac{1}{2}}.\end{aligned}$$

We square both sides of this inequality to see that

$$\begin{aligned}&\frac{1}{|B_{\frac{\rho}{2}}(y)|_\beta}\int_{B_{\frac{\rho}{2}}(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\\ &\leq \left(1+C\frac{\rho^{\frac{\gamma}{2}}}{R^{\frac{\gamma}{2}}}+C\frac{\rho^\gamma}{R^\gamma}\right)\frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\\ &\leq \left(1+C\frac{\rho^{\frac{\gamma}{2}}}{R^{\frac{\gamma}{2}}}\right)\frac{1}{|B_\rho(y)|_\beta}\int_{B_\rho(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\end{aligned}\tag{5.1.8}$$

This holds on every $B_\rho(y)$ with $B_\rho^+(y)\in\mathcal{B}^+(x_0,\theta_0 R,\frac{\theta_0 R}{2})$.

Now we iterate this estimate on concentric balls. Consider $B_{\frac{\theta_0 R}{2}}(y)$ with $B^+_{\frac{\theta_0 R}{2}}(y)\in\mathcal{B}^+(x_0,\theta_0 R,\frac{\theta_0 R}{2})$. Let $\rho_k=2^{-k}\frac{\theta_0 R}{2}$ for $k\in\mathbb{N}_0$. It follows from (5.1.8) that, for every $k\geq 1$, we have

$$\begin{aligned}&\frac{1}{|B_{\rho_k}(y)|_\beta}\int_{B_{\rho_k}(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\\ &\leq\left(1+C\frac{\rho_{k-1}^{\frac{\gamma}{2}}}{R^{\frac{\gamma}{2}}}\right)\frac{1}{|B_{\rho_{k-1}}(y)|_\beta}\int_{B_{\rho_{k-1}}(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x.\end{aligned}$$

Hence, applying this repeatedly gives

$$\begin{aligned}
&\frac{1}{|B_{\rho_k}(y)|_\beta}\int_{B_{\rho_k}(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\\
&\le \prod_{j=0}^{k-1}\left(1+C\frac{\rho_j^{\frac{\gamma}{2}}}{R^{\frac{\gamma}{2}}}\right)\frac{1}{|B_{\frac{\theta_0 R}{2}}(y)|_\beta}\int_{B_{\frac{\theta_0 R}{2}}(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\\
&\le \prod_{j=0}^{\infty}\left(1+C\frac{\rho_j^{\frac{\gamma}{2}}}{R^{\frac{\gamma}{2}}}\right)\frac{1}{|B_{\frac{\theta_0 R}{2}}(y)|_\beta}\int_{B_{\frac{\theta_0 R}{2}}(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x. \qquad (5.1.9)
\end{aligned}$$

We consider the infinite product in (5.1.9). Observe that $\frac{\rho_j^{\frac{\gamma}{2}}}{R^{\frac{\gamma}{2}}}\le 2^{-\frac{j\gamma}{2}}$. Hence

$$\prod_{j=0}^{\infty}\left(1+C\frac{\rho_j^{\frac{\gamma}{2}}}{R^{\frac{\gamma}{2}}}\right)\le\prod_{j=0}^{\infty}\left(1+C2^{-\frac{j\gamma}{2}}\right)\le\tilde{C}<\infty$$

where $\tilde{C}$ depends only on m, N, β. Thus it follows from (5.1.9) that, for every $k\in\mathbb{N}_0$, we have

$$\frac{1}{|B_{\rho_k}(y)|_\beta}\int_{B_{\rho_k}(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\le\tilde{C}\frac{1}{|B_{\frac{\theta_0 R}{2}}(y)|_\beta}\int_{B_{\frac{\theta_0 R}{2}}(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x. \qquad (5.1.10)$$

Furthermore, since $B_{\frac{\theta_0 R}{2}}(y)\subset B_{\theta_0 R}(x_0)$, we deduce that

$$\frac{1}{|B_{\frac{\theta_0 R}{2}}(y)|_\beta}\int_{B_{\frac{\theta_0 R}{2}}(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\le C\frac{1}{|B_{\theta_0 R}(x_0)|_\beta}\int_{B_{\theta_0 R}(x_0)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x. \qquad (5.1.11)$$

The combination of (5.1.10) and (5.1.11) implies

$$\begin{aligned}
\frac{1}{|B_{\rho_k}(y)|_\beta}\int_{B_{\rho_k}(y)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x &\le C\frac{1}{|B_{\theta_0 R}(x_0)|_\beta}\int_{B_{\theta_0 R}(x_0)}|x_{m+1}|^\beta|\nabla v|^2\mathrm{d}x\\
&= C\frac{1}{|B^+_{\theta_0 R}(x_0)|_\beta}\int_{B^+_{\theta_0 R}(x_0)}x_{m+1}^\beta|\nabla v|^2\mathrm{d}x
\end{aligned} \qquad (5.1.12)$$

for every $k\in\mathbb{N}_0$.

To conclude we show that we can use (5.1.12) to obtain a uniform bound for $||\nabla v||^2_{L^\infty(B_{\frac{y_{m+1}}{2\theta_1}}(y);\mathbb{R}^{(m+1)n})}$ where $B_{\frac{y_{m+1}}{\theta_1}}(y)\in\mathcal{B}_{\theta_1}(x_0,\theta_0 R,\frac{\theta_0 R}{3})$. Recall the inclusions given in (5.1.3). Using (5.1.2) and recalling the notation $y^+ = y-(0, y_{m+1})$, we calculate

$$
\begin{aligned}
||\nabla v||^2_{L^\infty(B_{\frac{y_{m+1}}{2\theta_1}}(y);\mathbb{R}^{(m+1)n})} &\leq C\frac{1}{|B_{\frac{y_{m+1}}{\theta_1}}(y)|}\int_{B_{\frac{y_{m+1}}{\theta_1}}(y)} |\nabla v|^2 \mathrm{d}x \\
&\leq C\frac{1}{|B^+_{\frac{(\theta_1+1)y_{m+1}}{\theta_1}}(y^+)|_\beta}\int_{B^+_{\frac{(\theta_1+1)y_{m+1}}{\theta_1}}(y^+)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x.
\end{aligned}
\tag{5.1.13}
$$

Now observe that

$$
\begin{aligned}
&\frac{1}{|B^+_{\frac{(\theta_1+1)y_{m+1}}{\theta_1}}(y^+)|_\beta}\int_{B^+_{\frac{(\theta_1+1)y_{m+1}}{\theta_1}}(y^+)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x \\
&\leq C\frac{1}{|B^+_{\rho_k}(y^+)|_\beta}\int_{B^+_{\rho_k}(y^+)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x \\
&\leq C\frac{1}{|B_{\rho_k}(y^+)|_\beta}\int_{B_{\rho_k}(y^+)} |x_{m+1}|^\beta|\nabla v|^2 \mathrm{d}x
\end{aligned}
\tag{5.1.14}
$$

where $\rho_k = \frac{\theta_0 R}{2}2^{-k}$ for some $k \in \mathbb{N}_0$. Together, (5.1.12), (5.1.13) and (5.1.14) yield

$$
||\nabla v||^2_{L^\infty(B_{\frac{y_{m+1}}{2\theta_1}}(y);\mathbb{R}^{(m+1)n})} \leq C\frac{1}{|B^+_{\theta_0 R}(x_0)|_\beta}\int_{B^+_{\theta_0 R}(x_0)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x. \tag{5.1.15}
$$

This holds for every $B_{\frac{y_{m+1}}{\theta_1}}(y) \in \mathcal{B}_{\theta_1}(x_0, \theta_0 R, \frac{\theta_0 R}{3})$. Finally we deduce that for almost every $y \in B^+_{\frac{\theta_0 R}{3}}(x_0)$ we have

$$
|\nabla v|^2(y) \leq C\frac{1}{|B^+_{\theta_0 R}(x_0)|_\beta}\int_{B^+_{\theta_0 R}(x_0)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x \tag{5.1.16}
$$

so that

$$
||\nabla v||^2_{L^\infty(B^+_{\frac{\theta_0 R}{3}}(x_0);\mathbb{R}^{(m+1)n})} \leq C\frac{1}{|B^+_{\theta_0 R}(x_0)|_\beta}\int_{B^+_{\theta_0 R}(x_0)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x. \tag{5.1.17}
$$

□

Remark 5.1.0.1. The choice of θ in the preceding lemma is $\frac{\theta_0}{3}$ where θ_0 is the number from Theorem 3.12.1.1 such that if $R^{1-m-\beta}\int_{B^+_R(x_0)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x \leq \varepsilon$ then $v \in C^{0,\gamma}(\overline{B^+_{\theta_0 R}(x_0)}; N)$.

We readily deduce from this lemma that v is Lipschitz continuous on $B^+_{\theta R}(x_0)$.

Corollary 5.1.0.1. *Suppose the assumptions of Lemma 5.1.0.1 hold. Then v is Lipschitz continuous in $B^+_{\theta R}(x_0)$. Furthermore, v satisfies*

$$|v(x) - v(y)| \leq C \left((3\theta R)^{1-m-\beta} \int_{B^+_{(3\theta R)}(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \right)^{\frac{1}{2}} \frac{|x-y|}{\theta R} \tag{5.1.18}$$

for every $x, y \in B^+_{\theta R}(x_0)$.

Proof. Assuming the hypotheses in Lemma 5.1.0.1 hold, we have

$$\begin{aligned} ||\nabla v||^2_{L^\infty(B^+_{\theta R}(x_0);\mathbb{R}^{(m+1)n})} &\leq C \frac{1}{|B^+_{3\theta R}(x_0)|_\beta} \int_{B^+_{3\theta R}(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \\ &= \frac{C}{\theta^2 R^2} (3\theta R)^{1-m-\beta} \int_{B^+_{3\theta R}(x_0)} x^\beta_{m+1} |\nabla v|^2 \mathrm{d}x \end{aligned}$$

for θ and C depending on m, N and β. Now, using the Mean Value Theorem, we see that

$$|v(x) - v(y)| \leq C ||\nabla v||_{L^\infty(B^+_{\theta R}(x_0);\mathbb{R}^{(m+1)n})} |x-y|$$

for $x, y \in B^+_{\theta R}(x_0)$. The combination of the two preceding inequalities implies (5.1.18). □

5.2 Existence of second order derivatives

The difference quotient method is an effective tool for establishing the existence of higher order derivatives in the theory of elliptic partial differential equations. It can be used to show that a continuous harmonic map is smooth [27]. We use difference quotients to show that the derivatives $\partial_i v$, with $i = 1, \ldots, m$, of a minimiser v of E^β relative to $\mathcal{O}$ are in $W^{1,2}_\beta$, taking advantage of the bound for the essential supremum of the gradient of v described in Section 5.1. We could however still obtain existence of these derivatives only using the Hölder continuity of v.

Lemma 5.2.0.1. *Let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ be a minimiser of E^β relative to $\mathcal{O}$ and let $B^+_R(x_0)$ be a half-ball with $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$. Suppose $v \in C^0(\overline{B^+_R(x_0)}; N) \cap W^{1,2}_\beta(B^+_R(x_0); N)$ and $\nabla v \in L^\infty(B^+_R(x_0); \mathbb{R}^{(m+1)n})$. Then, for $i = 1, \ldots, m$, the weak derivative $\nabla \partial_i v$ exists and satisfies*

$$\nabla \partial_i v \in L^2_\beta(B^+_{\frac{R}{2}}(x_0); \mathbb{R}^{(m+1)n}).$$

Proof. First we recall (3.1.6); for any $\phi \in C_0^\infty(B_R(x_0);\mathbb{R}^n)$ we have

$$\int_{B_R^+(x_0)} x_{m+1}^\beta \langle \nabla v, \nabla \phi \rangle \mathrm{d}x = \int_{B_R^+(x_0)} x_{m+1}^\beta \langle \phi, A(v)(\nabla v, \nabla v)\rangle \mathrm{d}x. \tag{5.2.1}$$

Let $\eta \in C_0^\infty(B_{\frac{3R}{4}}(x_0))$ be a smooth cutoff function such that $\eta \equiv 1$ in $B_{\frac{R}{2}}(x_0)$, $1 \geq \eta \geq 0$ in $B_{\frac{3R}{4}}(x_0)\backslash B_{\frac{R}{2}}(x_0)$ and $|\nabla \eta| \leq \frac{C}{R}$. Furthermore, let $\Delta_i^h v = h^{-1}(v(x+he_i) - v(x))$ be the difference quotient of v and assume $|h| < \frac{R}{4}$. It follows, using approximation, that $w = -\Delta_i^{-h}(\eta^2 \Delta_i^h v)$ is an admissible test function for (5.2.1); substituting w into (5.2.1) yields

$$\int_{B_R^+(x_0)} x_{m+1}^\beta \langle \nabla v, \nabla w \rangle \mathrm{d}x = \int_{B_R^+(x_0)} x_{m+1}^\beta \langle w, A(v)(\nabla v, \nabla v)\rangle \mathrm{d}x. \tag{5.2.2}$$

We consider each term in (5.2.2) separately. We have

$$\begin{aligned}
\int_{B_R^+(x_0)} x_{m+1}^\beta \langle \nabla v, \nabla w \rangle \mathrm{d}x &= \int_{B_R^+(x_0)} x_{m+1}^\beta \langle \nabla v, \nabla(-\Delta_i^{-h}(\eta^2 \Delta_i^h v))\rangle \mathrm{d}x \\
&= \int_{B_R^+(x_0)} x_{m+1}^\beta \langle \nabla v, -\Delta_i^{-h} \nabla(\eta^2 \Delta_i^h v)\rangle \mathrm{d}x \\
&= \int_{B_R^+(x_0)} x_{m+1}^\beta \langle \Delta_i^h \nabla v, \nabla(\eta^2 \Delta_i^h v)\rangle \mathrm{d}x \\
&= \int_{B_R^+(x_0)} 2\eta x_{m+1}^\beta \langle \Delta_i^h \nabla v \cdot \nabla \eta, \Delta_i^h v)\rangle \mathrm{d}x \\
&+ \quad \int_{B_R^+(x_0)} \eta^2 x_{m+1}^\beta |\Delta_i^h \nabla v|^2 \mathrm{d}x.
\end{aligned} \tag{5.2.3}$$

Furthermore, an integration by parts in the term involving A gives

$$\int_{B_R^+(x_0)} x_{m+1}^\beta \langle w, A(v)(\nabla v, \nabla v)\rangle \mathrm{d}x = \int_{B_R^+(x_0)} x_{m+1}^\beta \langle \eta^2 \Delta_i^h v, \Delta_i^h (A(v)(\nabla v, \nabla v))\rangle \mathrm{d}x. \tag{5.2.4}$$

Recalling that the support of η is contained in $B_{\frac{3R}{4}}(x_0)$, we combine (5.2.2), (5.2.3) and (5.2.4) to see that

$$\begin{aligned}
\int_{B_R^+(x_0)} \eta^2 x_{m+1}^\beta |\Delta_i^h \nabla v|^2 \mathrm{d}x &= \int_{B_{\frac{3R}{4}}^+(x_0)} \eta^2 x_{m+1}^\beta \langle \Delta_i^h v, \Delta_i^h (A(v)(\nabla v, \nabla v))\rangle \mathrm{d}x \\
&- \int_{B_{\frac{3R}{4}}^+(x_0)} 2\eta x_{m+1}^\beta \langle \Delta_i^h \nabla v \cdot \nabla \eta, \Delta_i^h v\rangle \mathrm{d}x.
\end{aligned} \tag{5.2.5}$$

We now use Young's inequality, $ab \leq \frac{a^2}{\delta 2} + \delta \frac{b^2}{2}$ for $a, b \geq 0$ and $\delta > 0$, to move all of the terms involving $\Delta_i^h \nabla v$ on the right hand side of (5.2.5) to the left hand side. We calculate

$$-\int_{B^+_{\frac{3R}{4}}(x_0)} 2\eta x_{m+1}^\beta \langle \Delta_i^h \nabla v \cdot \nabla \eta, \Delta_i^h v\rangle \mathrm{d}x \leq C\delta \int_{B^+_R(x_0)} \eta^2 x_{m+1}^\beta |\Delta_i^h \nabla v|^2 \mathrm{d}x + \frac{C}{\delta}\int_{B^+_R(x_0)} x_{m+1}^\beta |\nabla \eta|^2 |\Delta_i^h v|^2 \mathrm{d}x. \tag{5.2.6}$$

We need to estimate the term involving $\Delta_i^h(A(v)(\nabla v, \nabla v))$ in a similar fashion. To this end, without relabelling we extend A to a section of $T^*\mathbb{R}^n \otimes T^*\mathbb{R}^n \otimes T\mathbb{R}^n$ and write $x_h = x + he_i$. The integrals in (5.2.5) all vanish outside $B^+_{\frac{3R}{4}}(x_0)$ so we assume $x \in B^+_{\frac{3R}{4}}(x_0)$. Notice that the Mean Value Theorem yields

$$|\Delta_i^h v(x)| \leq ||\nabla v||_{L^\infty(B^+_R(x_0);\mathbb{R}^{(m+1)n})}. \tag{5.2.7}$$

Now we calculate

$$\begin{aligned}
&\Delta_i^h(A(v)(\nabla v, \nabla v))(x)\\
&= h^{-1}(A(v(x_h))(\nabla v(x_h), \nabla v(x_h)) - A(v(x))(\nabla v(x_h), \nabla v(x_h)))\\
&\quad + A(v(x))(\Delta_i^h \nabla v, \nabla v(x_h)) + A(v(x))(\nabla v, \Delta_i^h \nabla v).
\end{aligned}$$

Consequently, an application of the Mean Value Theorem and (5.2.7) gives

$$\begin{aligned}
\left|\Delta_i^h(A(v)(\nabla v, \nabla v))(x)\right| &\leq C|\Delta_i^h \nabla v| ||\nabla v||_{L^\infty(B^+_R(x_0);\mathbb{R}^{(m+1)n})}\\
&\quad + C||\nabla v||^3_{L^\infty(B^+_R(x_0);\mathbb{R}^{(m+1)n})}.
\end{aligned} \tag{5.2.8}$$

Together, (5.2.7) and (5.2.8) imply

$$\begin{aligned}
&\int_{B^+_{\frac{3R}{4}}(x_0)} \eta^2 x_{m+1}^\beta \langle \Delta_i^h v, \Delta_i^h(A(v)(\nabla v, \nabla v))\rangle \mathrm{d}x\\
&\leq C\int_{B^+_{\frac{3R}{4}}(x_0)} \eta^2 x_{m+1}^\beta |\Delta_i^h \nabla v| ||\nabla v||^2_{L^\infty(B^+_R(x_0);\mathbb{R}^{(m+1)n})} \mathrm{d}x\\
&\quad + C||\nabla v||^4_{L^\infty(B^+_R(x_0);\mathbb{R}^{(m+1)n})} \int_{B^+_{\frac{3R}{4}}(x_0)} x_{m+1}^\beta \mathrm{d}x.
\end{aligned} \tag{5.2.9}$$

An application of Young's inequality gives

$$\begin{aligned}
&\int_{B^+_{\frac{3R}{4}}(x_0)} \eta^2 x_{m+1}^\beta |\Delta_i^h \nabla v| ||\nabla v||^2_{L^\infty(B^+_R(x_0);\mathbb{R}^{(m+1)n})} \mathrm{d}x \\
&\leq C\delta \int_{B^+_{\frac{3R}{4}}(x_0)} \eta^2 x_{m+1}^\beta |\Delta_i^h \nabla v|^2 \mathrm{d}x \\
&+ \frac{C}{\delta} \int_{B^+_{\frac{3R}{4}}(x_0)} \eta^2 x_{m+1}^\beta ||\nabla v||^4_{L^\infty(B^+_R(x_0);\mathbb{R}^{(m+1)n})} \mathrm{d}x.
\end{aligned} \tag{5.2.10}$$

Choosing δ sufficiently small in (5.2.6) and (5.2.10) we combine these inequalities with (5.2.5) and (5.2.9). This yields

$$\begin{aligned}
&\int_{B^+_R(x_0)} \eta^2 x_{m+1}^\beta |\Delta_i^h \nabla v|^2 \mathrm{d}x \\
&\leq C||\nabla v||^4_{L^\infty(B^+_R(x_0);\mathbb{R}^{(m+1)n})} \int_{B^+_R(x_0)} x_{m+1}^\beta \mathrm{d}x + C \int_{B^+_R(x_0)} x_{m+1}^\beta |\nabla \eta|^2 |\Delta_i^h v|^2 \mathrm{d}x.
\end{aligned} \tag{5.2.11}$$

Since $|\nabla \eta| \leq \frac{C}{R}$ and $\eta \equiv 1$ in $B^+_{\frac{R}{2}}(x_0)$, using (5.2.7) we have

$$\begin{aligned}
&\int_{B^+_{\frac{R}{2}}(x_0)} x_{m+1}^\beta |\Delta_i^h \nabla v|^2 \mathrm{d}x \\
&\leq C||\nabla v||^2_{L^\infty(B^+_R(x_0);\mathbb{R}^{(m+1)n})} (R^{-2} + ||\nabla v||^2_{L^\infty(B^+_R(x_0);\mathbb{R}^{(m+1)n})}) \int_{B^+_R(x_0)} x_{m+1}^\beta \mathrm{d}x.
\end{aligned}$$

We observe that this bound is independent of h with $|h| < \frac{R}{4}$. Hence by Lemma 4.1.0.2 we conclude that the weak derivative $\nabla \partial_i v$ exists and satisfies the above inequality with $\nabla \partial_i v$ in place of $\Delta_i^h \nabla v$. This concludes the proof. $\square$

5.3 Caccioppoli-Type Inequality

Here we show a Caccioppoli-type estimate for the derivatives $\partial_i v$, with $i = 1, \ldots, m$, of a minimiser v of E^β relative to $\mathcal{O}$. Such inequalities are widely used in the theory of elliptic partial differential equations; they capitalise on the ellipticity in order to obtain control of higher order derivatives in terms of lower order derivatives.

Lemma 5.3.0.1. *Let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ be a minimiser of E^β relative to $\mathcal{O}$ and let $B^+_R(x_0)$ be a half-ball with $R \leq 1$ and $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$. Suppose*

$v \in C^0(\overline{B_R^+(x_0)}; N)$, $\nabla v \in L^\infty(B_R^+(x_0); \mathbb{R}^{(m+1)n})$ and $\partial_i v \in W_\beta^{1,2}(B_R^+(x_0); \mathbb{R}^n)$ for some $i = 1, \ldots, m$. Let $B_\rho(y) \subset B_R(x_0)$ with $y_{m+1} \geq 0$. Then there is a constant $C = C(m, N, \beta)$ such that

$$\begin{aligned}
&\int_{B_{\frac{\rho}{2}}(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \left|\nabla\partial_i v\right|^2 \mathrm{d}x \\
&\leq C\left(||\nabla v||^2_{L^\infty(B_\rho(y)\cap\mathbb{R}^{m+1}_+;\mathbb{R}^{(m+1)n})} + \frac{1}{\rho^2}\right)\int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \left|\partial_i v - \lambda\right|^2 \mathrm{d}x \\
&\quad + C||\nabla v||^6_{L^\infty(B_\rho(y)\cap\mathbb{R}^{m+1}_+;\mathbb{R}^{(m+1)n})}\int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \mathrm{d}x
\end{aligned} \tag{5.3.1}$$

for any $\lambda \in \mathbb{R}^n$.

Proof. Integrating by parts with respect to x_i in (3.1.6) shows that for every $\psi \in W_{\beta,0}^{1,2}(B_R(x_0); \mathbb{R}^n)$, we have

$$\begin{aligned}
&\int_{B_R^+(x_0)} x_{m+1}^\beta \left\langle \nabla\partial_i v, \nabla\psi\right\rangle \mathrm{d}x \\
&= \int_{B_R^+(x_0)} x_{m+1}^\beta \left\langle \psi, 2A(v)\left(\nabla\partial_i v, \nabla v\right) + DA(v)\left(\nabla v, \nabla v, \partial_i v\right)\right\rangle \mathrm{d}x
\end{aligned}$$

where DA represents the derivative of $A(y)(\cdot,\cdot)$ with respect to y. Now, choose $\psi = \eta^2\left(\partial_i v - \lambda\right)$ where $\lambda \in \mathbb{R}^n$ is a constant vector and $\eta \in C_0^\infty(B_\rho(y))$ is a cutoff function with $\eta \equiv 1$ in $B_{\frac{\rho}{2}}(y)$, $0 \leq \eta \leq 1$, and $|\nabla\eta| \leq \frac{C}{\rho}$. We calculate

$$\begin{aligned}
\int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta\eta^2 \left|\nabla\partial_i v\right|^2 \mathrm{d}x \leq{}& C\int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta\eta^2 \left|\partial_i v - \lambda\right| \left|\nabla\partial_i v\right| \left|\nabla v\right| \mathrm{d}x \\
&+ C\int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta\eta^2 \left|\partial_i v - \lambda\right| \left|\partial_i v\right| \left|\nabla v\right|^2 \mathrm{d}x \\
&+ C\int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta\eta|\nabla\eta| \left|\partial_i v - \lambda\right| \left|\nabla\partial_i v\right| \mathrm{d}x.
\end{aligned} \tag{5.3.2}$$

We consider each term on the right hand side of (5.3.2) separately, applying Young's inequalitiy, $ab \leq \frac{\delta a^2}{2} + \frac{b^2}{\delta 2}$ for $a, b \geq 0$ and $\delta > 0$, to each in turn. Since

$||\nabla v||_{L^\infty(B_R^+(x_0);\mathbb{R}^{(m+1)n})} < \infty$, we calculate

$$\begin{aligned}
&\int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \eta^2 \,|\partial_i v - \lambda|\, |\nabla \partial_i v|\, |\nabla v| \,\mathrm{d}x \\
&\leq \frac{C}{\delta} ||\nabla v||^2_{L^\infty(B_\rho(y)\cap\mathbb{R}^{m+1}_+;\mathbb{R}^{(m+1)n})} \int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \eta^2 \,|\partial_i v - \lambda|^2 \,\mathrm{d}x \\
&\quad + C\delta \int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \eta^2 \,|\nabla \partial_i v|^2 \,\mathrm{d}x, \qquad (5.3.3)
\end{aligned}$$

$$\begin{aligned}
&\int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \eta^2 \,|\partial_i v - \lambda|\, |\partial_i v|\, |\nabla v|^2 \,\mathrm{d}x \\
&\leq C \int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \,|\partial_i v - \lambda|^2 \,\mathrm{d}x \\
&\quad + C||\nabla v||^6_{L^\infty(B_\rho(y)\cap\mathbb{R}^{m+1}_+;\mathbb{R}^{(m+1)n})} \int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \mathrm{d}x \qquad (5.3.4)
\end{aligned}$$

and, using additionally the fact that $|\nabla\eta| \leq \frac{C}{\rho}$,

$$\begin{aligned}
&\int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \eta |\nabla\eta| \,|\partial_i v - \lambda|\, |\nabla \partial_i v| \,\mathrm{d}x \\
&\leq \frac{1}{\delta}\frac{C}{\rho^2} \int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \,|\partial_i v - \lambda|^2 \,\mathrm{d}x \\
&\quad C\delta \int_{B_\rho(y)\cap\mathbb{R}^{m+1}_+} x_{m+1}^\beta \eta^2 \,|\nabla \partial_i v|^2 \,\mathrm{d}x. \qquad (5.3.5)
\end{aligned}$$

The combination of (5.3.2) with (5.3.3), (5.3.4) and (5.3.5) yield (5.3.1) provided δ is chosen sufficiently small in (5.3.3) and (5.3.5). □

5.4 Improved Control in the Poincaré Inequality for First Derivatives on the Boundary

In Section 3.10 we showed that if the energy of a minimiser of E^β relative to $\mathcal{O}$ is sufficiently small, then we obtain improved control in the Poincaré inequality, provided the integrals are multiplied by factors making them scaling invariant. We can obtain a similar improvement in the Poincaré inequality for the derivatives $\partial_i v$, where $i = 1, \ldots, m$, of a minimiser v of E^β relative to $\mathcal{O}$, still only assuming the energy of v is sufficiently small, by taking advantage of the Caccioppoli inequality, Lemma 5.3.0.1.

Lemma 5.4.0.1. *Let $i = 1, \ldots, m$. For every $\delta > 0$ there exist numbers $\varepsilon > 0$, $\tau \in (0, 1)$ and $\theta \in (0, \frac{1}{4}]$ such that the following holds. Suppose $v \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_{+}; N)$ is a minimiser of E^{β} relative to $\mathcal{O}$. Let $B^{+}_{R}(x_0) \subset \mathbb{R}^{m+1}_{+}$ be a half-ball with $R \leq 1$ and $\overline{\partial^0 B^{+}_{R}(x_0)} \subset \mathcal{O}$. If*

$$R^{1-m-\beta} \int_{B^{+}_{R}(x_0)} x^{\beta}_{m+1} |\nabla v|^2 \, \mathrm{d}x \leq \varepsilon^2,$$

then, for every $B^{+}_{r}(y) \in \mathcal{B}^{+}(x_0, R, \tau R)$, either

$$r^{1-m-\beta} \int_{B^{+}_{r}(y)} x^{\beta}_{m+1} |\nabla \partial_i v|^2 \, \mathrm{d}x \leq \delta \frac{1}{R^2} \left(\frac{r}{R}\right)^2 \tag{5.4.1}$$

or

$$(\theta r)^{-(1+m+\beta)} \int_{B^{+}_{\theta r}(y)} x^{\beta}_{m+1} \left|\partial_i v - \overline{\partial_i v}_{B^{+}_{\theta r}(y),\beta}\right|^2 \mathrm{d}x \leq \delta r^{1-m-\beta} \int_{B^{+}_{r}(y)} x^{\beta}_{m+1} |\nabla \partial_i v|^2 \, \mathrm{d}x. \tag{5.4.2}$$

Proof. First we observe that the statement of the lemma is invariant under rescaling and translation by any point in $\partial\mathbb{R}^{m+1}_{+}$. We will show that if the lemma is true on $B^{+}_{1}(0)$ for minimisers of E^{β} relative to $\tilde{\mathcal{O}}$ whenever $\overline{\partial^0 B^{+}_{1}(0)} \subset \tilde{\mathcal{O}}$, then we may obtain the lemma on $B^{+}_{R}(x_0)$ for a minimiser of E^{β} relative to $\mathcal{O}$ whenever $\overline{\partial^0 B^{+}_{R}(x_0)} \subset \mathcal{O}$ and $R \leq 1$. To achieve this we will use rescaling via the map $x \mapsto Rx + x_0$, defined for $x \in B^{+}_{1}(0)$, and apply the lemma on $B^{+}_{1}(0)$ to the map $v_R(x) = v(Rx + x_0)$.

Suppose the lemma holds on $B^{+}_{1}(0)$ for a minimiser of E^{β} relative to $\tilde{\mathcal{O}}$ whenever $\overline{\partial^0 B^{+}_{1}(0)} \subset \tilde{\mathcal{O}}$. Let $v \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_{+}; N)$ be a minimising harmonic map relative to $\mathcal{O}$, let $B^{+}_{R}(x_0)$ satisfy $\overline{\partial^0 B^{+}_{R}(x_0)} \subset \mathcal{O}$ and suppose

$$R^{1-m-\beta} \int_{B^{+}_{R}(x_0)} x^{\beta}_{m+1} |\nabla v|^2 \, \mathrm{d}x \leq \varepsilon^2.$$

Define $v_R(\cdot) = v(R \cdot + x_0) \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_{+}; N)$. Using the change of variables $x \mapsto Rx + x_0$ we see that

$$\int_{B^{+}_{1}(0)} x^{\beta}_{m+1} |\nabla v_R|^2 \, \mathrm{d}x = R^{1-m-\beta} \int_{B^{+}_{R}(x_0)} x^{\beta}_{m+1} |\nabla v|^2 \, \mathrm{d}x \leq \varepsilon^2.$$

Now we show v_R is a minimiser relative to $\tilde{\mathcal{O}} := \{\frac{x - x_0}{R} : x \in \mathcal{O}\}$. Since $B^{+}_{R}(x_0)$ satisfies $\overline{\partial^0 B^{+}_{R}(x_0)} \subset \mathcal{O}$, it follows from the definition of $\tilde{\mathcal{O}}$ that $B^{+}_{1}(0)$ satisfies $\overline{\partial^0 B^{+}_{1}(0)} \subset \tilde{\mathcal{O}}$. Let $\tilde{w} \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_{+}; N)$ with $\tilde{w}|_{\mathbb{R}^{m+1}_{+} \setminus \tilde{K}} = v_R|_{\mathbb{R}^{m+1}_{+} \setminus \tilde{K}}$ for a compact $\tilde{K} \subset \mathbb{R}^{m+1}$ with $\tilde{K} \cap \partial\mathbb{R}^{m+1}_{+} \subset \tilde{\mathcal{O}}$. Then $w(\cdot) = \tilde{w}\left(\frac{\cdot - x_0}{R}\right) \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_{+}; N)$ and

$w|_{\mathbb{R}^{m+1}_+\setminus K} = v|_{\mathbb{R}^{m+1}_+\setminus K}$ where K is the image of $\tilde{K}$ under the change of variables $x \mapsto Rx + x_0$. Since v is a minimiser, we calculate

$$\begin{aligned}
\int_{\mathbb{R}^{m+1}_+} x_{m+1}^\beta \left|\nabla v_R\right|^2 \mathrm{d}x &= R^{1-m-\beta} \int_{\mathbb{R}^{m+1}_+} x_{m+1}^\beta \left|\nabla v\right|^2 \mathrm{d}x \\
&\le R^{1-m-\beta} \int_{\mathbb{R}^{m+1}_+} x_{m+1}^\beta \left|\nabla w\right|^2 \mathrm{d}x \\
&= R^{-(1+m+\beta)} \int_{\mathbb{R}^{m+1}_+} x_{m+1}^\beta \left|\nabla \tilde{w}\left(\frac{x-x_0}{R}\right)\right|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^{m+1}_+} x_{m+1}^\beta \left|\nabla \tilde{w}\right|^2 \mathrm{d}x.
\end{aligned}$$

Hence v_R is a minimiser of E^β relative to $\tilde{\mathcal{O}}$. Thus the conclusions of the lemma hold for v_R on $B_1^+(0)$ by assumption. This implies that for every $B_s^+(z) \in \mathcal{B}^+(0,1,\tau)$ either

$$s^{1-m-\beta} \int_{B_s^+(z)} x_{m+1}^\beta \left|\nabla \partial_i(v_R)\right|^2 \mathrm{d}x \le \delta s^2 \tag{5.4.3}$$

or

$$\begin{aligned}
&(\theta s)^{-(1+m+\beta)} \int_{B_{\theta s}^+(z)} x_{m+1}^\beta \left|\partial_i(v_R) - \overline{\partial_i(v_R)}_{B_{\theta s}^+(z),\beta}\right|^2 \mathrm{d}x \\
&\le \delta s^{1-m-\beta} \int_{B_s^+(z)} x_{m+1}^\beta \left|\nabla \partial_i(v_R)\right|^2 \mathrm{d}x.
\end{aligned} \tag{5.4.4}$$

We want to show (5.4.1) and (5.4.2) hold for v. Notice that $B_r^+(y) \in \mathcal{B}^+(x_0, R, \tau R)$ if, and only if, $B_s^+(z) \in \mathcal{B}^+(0,1,\tau)$ where s and r are related by $s = \frac{r}{R}$ and y and z are related by $z = \frac{y-x_0}{R}$. Furthermore, $B_s(z)$ is the image of $B_r(y)$ under the change of variables $x \mapsto \frac{x-x_0}{R}$.

First we show (5.4.3) implies (5.4.1). Suppose r, s, y and z are related as described previously. We calculate

$$\begin{aligned}
&s^{1-m-\beta} \int_{B_s^+(z)} x_{m+1}^\beta \left|\nabla \partial_i(v_R)\right|^2 \mathrm{d}x \\
&= R^4 s^{1-m-\beta} \int_{B_s^+(z)} x_{m+1}^\beta \left|\nabla \partial_i v(Rx+x_0)\right|^2 \mathrm{d}x \\
&= R^2 (Rs)^{1-m-\beta} \int_{B_s^+(z)} x_{m+1}^\beta \left|\nabla \partial_i v(Rx+x_0)\right|^2 R^{1+m+\beta} \mathrm{d}x \\
&= R^2 r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta \left|\nabla \partial_i v\right|^2 \mathrm{d}x.
\end{aligned} \tag{5.4.5}$$

Hence, if (5.4.3) holds, we deduce from (5.4.5) that

$$R^2 r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta \left|\nabla \partial_i v\right|^2 \mathrm{d}x \leq \delta \left(\frac{r}{R}\right)^2$$

which implies (5.4.1).

Now we show (5.4.2) follows from (5.4.4). Using the change of variables $x \mapsto Rx + x_0$ we see that $\overline{\partial_i(v_R)}_{B_{\theta s}^+(z),\beta} = R\overline{\partial_i v}_{B_{\theta r}^+(y),\beta}$. Hence, using the same change of variables again, we have

$$\begin{aligned}
&(\theta s)^{-(1+m+\beta)} \int_{B_{\theta s}^+(z)} x_{m+1}^\beta \left|\partial_i(v_R) - \overline{\partial_i(v_R)}_{B_{\theta s}^+(z),\beta}\right|^2 \mathrm{d}x \\
&= R^2(\theta s)^{-(1+m+\beta)} \int_{B_{\theta s}^+(z)} x_{m+1}^\beta \left|\partial_i v(Rx + x_0) - \overline{\partial_i v}_{B_{\theta r}^+(y),\beta}\right|^2 \mathrm{d}x \\
&= R^{1-m-\beta}(\theta s)^{-(1+m+\beta)} \int_{B_{\theta s}^+(z)} x_{m+1}^\beta \left|\partial_i v(Rx + x_0) - \overline{\partial_i v}_{B_{\theta r}^+(y),\beta}\right|^2 R^{(1+m+\beta)}\mathrm{d}x \\
&= R^2(\theta r)^{-(1+m+\beta)} \int_{B_{\theta r}^+(y)} x_{m+1}^\beta \left|\partial_i v - \overline{\partial_i v}_{B_{\theta r}^+(y),\beta}\right|^2 \mathrm{d}x. \qquad (5.4.6)
\end{aligned}$$

The combination of (5.4.5) and (5.4.6) yields (5.4.2). Thus if the lemma holds on $B_1^+(0)$ and the hypothesis of the lemma hold on $B_R^+(x_0)$ then the conclusion also holds on $B_R^+(x_0)$.

Henceforth we will assume $R = 1$, $x_0 = 0$ and v is a minimiser of E^β relative to $\mathcal{O}$ and $\overline{\partial^0 B_1^+(0)} \subset \mathcal{O}$. We argue by contradiction. In particular, we will show that if the lemma were false, then we may construct a weak solution of (2.4.5), that is a weak solution of $\operatorname{div}(x_{m+1}^\beta \nabla w) = 0$ in $B_1^+(0)$ with $x_{m+1}^\beta \frac{\partial w}{\partial x_{m+1}} = 0$ on $\partial^0 B_1^+(0)$, whose $L^2_\beta(B_1^+(0);\mathbb{R}^n)$ norm is bounded below and strictly above by the same number, a contradiction.

Suppose the statement is false. Then there exists a $\delta > 0$ such that, for any fixed $\theta \in (0, \frac{1}{4}]$ we may find a sequence $(v_k)_{k\in\mathbb{N}}$ of minimising harmonic maps, relative to $\mathcal{O}$, with $v_k \in W^{1,2}_\beta(B_1^+(0); N)$ such that

$$\int_{B_1^+(0)} x_{m+1}^\beta \left|\nabla v_k\right|^2 \mathrm{d}x := \varepsilon_k^2 \to 0,$$

and, furthermore, a sequence of numbers $0 < \tau_k \to 0^+$, half-balls $B_{r_k}^+(y_k) \in \mathcal{B}^+(0, 1, \tau_k)$, and numbers $0 < r_k \leq \tau_k \to 0^+$ such that

$$r_k^{1-m-\beta} \int_{B_{r_k}^+(y_k)} x_{m+1}^\beta \left|\nabla \partial_i v_k\right|^2 \mathrm{d}x > \delta r_k^2 \qquad (5.4.7)$$

and

$$(\theta r_k)^{-(1+m+\beta)} \int_{B^+_{\theta r_k}(y_k)} x^\beta_{m+1} \left|\partial_i v_k - \overline{(\partial_i v_k)}_{B^+_{\theta r_k}(y_k),\beta}\right|^2 \mathrm{d}x$$
$$> \delta r_k^{1-m-\beta} \int_{B^+_{r_k}(y_k)} x^\beta_{m+1} \left|\nabla \partial_i v_k\right|^2 \mathrm{d}x. \tag{5.4.8}$$

We show that $r_k^{1-m-\beta} \int_{B^+_{r_k}(y_k)} x^\beta_{m+1} \left|\nabla \partial_i v_k\right|^2 \mathrm{d}x \to 0$. Discarding as many v_k as necessary and re-indexing the resulting sequence, we may suppose that $\varepsilon_k^2 \leq \varepsilon$ for every k, where ε is the number from Theorem 3.12.1.1. Furthermore we may choose this sequence such that $2\tau_k \leq \frac{\theta_0}{12}$, where θ_0 is the number given by Theorem 3.12.1.1, so that $B^+_{r_k}(y_k) \subset B^+_{2r_k}(y_k) \subset B^+_{2\tau_k}(y_k) \in \mathcal{B}^+(0, \frac{\theta_0}{6}, \frac{\theta_0}{12})$ and, in particular, $B^+_{2\tau_k}(y_k) \subset B^+_{\frac{\theta_0}{6}}(0)$. Applying Theorem 5.1.0.1 we see that each v_k satisfies

$$||\nabla v_k||^2_{L^\infty(B^+_{\frac{\theta_0}{3}}(0);\mathbb{R}^{(m+1)n})} \leq \frac{C}{\theta_0^2} \theta_0^{1-m-\beta} \int_{B^+_{\theta_0}(0)} x^\beta_{m+1} |\nabla v_k|^2 \mathrm{d}x.$$

Furthermore, the boundary monotonicity formula, Lemma 3.3.1.1, implies that

$$\theta_0^{1-m-\beta} \int_{B^+_{\theta_0}(0)} x^\beta_{m+1} |\nabla v_k|^2 \mathrm{d}x \leq \int_{B^+_1(0)} x^\beta_{m+1} \left|\nabla v_k\right|^2 \mathrm{d}x = \varepsilon_k^2.$$

Thus,

$$||\nabla v_k||^2_{L^\infty(B^+_{\frac{\theta_0}{3}}(0);\mathbb{R}^{(m+1)n})} \leq \frac{C}{\theta_0^2} \varepsilon_k^2 \to 0. \tag{5.4.9}$$

Now we make use of the Caccioppoli-type inequality, Lemma 5.3.0.1, with $\lambda = 0$, noting that by Lemma 5.2.0.1 we have $\nabla \partial_i v_k \in L^2_\beta(B^+_{\frac{\theta_0}{6}}(0);\mathbb{R}^{(m+1)n})$ for $i = 1, \ldots, m$. We apply Lemma 5.3.0.1 on $B^+_{2\tau_k}(y_k) \subset B^+_{\frac{\theta_0}{6}}(0)$. This yields

$$\int_{B^+_{r_k}(y_k)} x^\beta_{m+1} \left|\nabla \partial_i v_k\right|^2 \mathrm{d}x$$
$$\leq C \left(||\nabla v_k||^2_{L^\infty(B^+_{2r_k}(y_k);\mathbb{R}^{(m+1)n})} + \frac{1}{r_k^2} \right) \int_{B^+_{2r_k}(y_k)} x^\beta_{m+1} \left|\partial_i v_k\right|^2 \mathrm{d}x$$
$$+ C||\nabla v_k||^6_{L^\infty(B^+_{2r_k}(y_k);\mathbb{R}^{(m+1)n})} \int_{B^+_{2r_k}(y_k)} x^\beta_{m+1} \mathrm{d}x \tag{5.4.10}$$

Since $||\nabla v_k||_{L^\infty(B^+_{2r_k}(y_k);\mathbb{R}^{(m+1)n})} \leq ||\nabla v_k||_{L^\infty(B^+_{\frac{\theta_0}{3}}(0);\mathbb{R}^{(m+1)n})}$ we may combine (5.4.9) with (5.4.10), multiplying the latter by $r_k^{1-m-\beta}$, to see that

$$\begin{aligned}
&r_k^{1-m-\beta}\int_{B^+_{r_k}(y_k)} x_{m+1}^\beta \left|\nabla\partial_i v_k\right|^2 \mathrm{d}x \\
&\le Cr_k^2||\nabla v_k||^4_{L^\infty(B^+_{\frac{\theta_0}{3}}(0);\mathbb{R}^{(m+1)n})} + C||\nabla v_k||^2_{L^\infty(B^+_{\frac{\theta_0}{3}}(0);\mathbb{R}^{(m+1)n})} \\
&\quad + Cr_k^2||\nabla v_k||^6_{L^\infty(B^+_{\frac{\theta_0}{3}}(0);\mathbb{R}^{(m+1)n})} \\
&\le C(r_k^2\varepsilon_k^4 + \varepsilon_k^2 + r_k^2\varepsilon_k^6) \to 0, \qquad (5.4.11)
\end{aligned}$$

where C depends on m, N, β and θ_0 and hence only on m, N, β. From (5.4.11), we conclude that

$$r_k^{1-m-\beta}\int_{B^+_{r_k}(y_k)} x_{m+1}^\beta \left|\nabla\partial_i v_k\right|^2 \mathrm{d}x := \tilde{\varepsilon}_k^2 \to 0. \qquad (5.4.12)$$

We see from (5.4.7) and (5.4.8) that

$$\tilde{\varepsilon}_k^2 > \delta r_k^2 \qquad (5.4.13)$$

and

$$(\theta r_k)^{-(1+m+\beta)}\int_{B^+_{\theta r_k}(y_k)} x_{m+1}^\beta \left|\partial_i v_k - \overline{(\partial_i v_k)}_{B^+_{\theta r_k}(y_k),\beta}\right|^2 \mathrm{d}x > \delta\tilde{\varepsilon}_k^2. \qquad (5.4.14)$$

In order to construct a weak solution of the Neumann type problem (2.4.5) with the desired properties, we define the normalised sequence

$$w_k = \frac{\partial_i v_k(r_k x + y_k) - \overline{(\partial_i v_k)}_{B^+_{\theta r_k}(y_k),\beta}}{\tilde{\varepsilon}_k}$$

and analyse the limit as $k \to \infty$. We calculate

$$\nabla w_k(x) = \frac{r_k}{\tilde{\varepsilon}_k}\nabla\partial_i v_k(r_k x + y_k). \qquad (5.4.15)$$

Hence, using the change of variables $x \mapsto r_k x + y_k$, we find

$$\int_{B^+_1(0)} x_{m+1}^\beta \left|\nabla w_k\right|^2 \mathrm{d}x = 1 \qquad (5.4.16)$$

and

$$\overline{(w_k)}_{B^+_\theta(0),\beta} = \frac{1}{\int_{B^+_\theta(0)} x_{m+1}^\beta \mathrm{d}x}\int_{B^+_\theta(0)} x_{m+1}^\beta w_k \mathrm{d}x = 0. \qquad (5.4.17)$$

Furthermore, after changing variables again, we deduce from (5.4.14) that

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |w_k|^2 \,\mathrm{d}x > \delta. \tag{5.4.18}$$

Using (5.4.17) we write $w_k = w_k - \overline{(w_k)}_{B_\theta^+(0),\beta}$ and an application of Lemma 2.3.3.4 shows that

$$\begin{aligned}\int_{B_1^+(0)} x_{m+1}^\beta |w_k|^2 \mathrm{d}x &= \int_{B_1^+(0)} x_{m+1}^\beta |w_k - \overline{(w_k)}_{B_\theta^+(0),\beta}|^2 \mathrm{d}x \\ &\leq C\theta^{-(1+m+\beta)} \int_{B_1^+(0)} x_{m+1}^\beta |\nabla w_k|^2 \,\mathrm{d}x. \end{aligned} \tag{5.4.19}$$

Together, (5.4.16) and (5.4.19) show that $(w_k)_{k\in\mathbb{N}}$ is bounded $W_\beta^{1,2}(B_1^+(0);\mathbb{R}^n)$. Hence, the the Rellich Compactness lemma, Lemma 2.3.5.2, yields a subsequence $(w_{k_j})_{j\in\mathbb{N}}$ which converges weakly in $W_\beta^{1,2}(B_1^+(0);\mathbb{R}^n)$ and strongly in $L_\beta^2(B_1^+(0);\mathbb{R}^n)$ to some $w \in W_\beta^{1,2}(B_1^+(0);\mathbb{R}^n)$.

Now we show that w is a weak solution of the Neumann-type problem 2.4.5 in $B_1^+(0)$. Let $\phi \in C_0^\infty(B_1(0);\mathbb{R}^n)$ and define $\tilde{\phi} \in C_0^\infty(B_{r_k}(y_k);\mathbb{R}^n)$ by $\tilde{\phi}(z) = \phi\left(\frac{z-y_k}{r_k}\right)$. We observe that

$$\nabla\tilde{\phi}(z) = \nabla\left(\phi\left(\frac{z-y_k}{r_k}\right)\right) = \frac{1}{r_k}\nabla\phi\left(\frac{z-y_k}{r_k}\right) = \frac{1}{r_k}\nabla\phi(x) \tag{5.4.20}$$

where $x \in B_1(0)$ and $z \in B_{r_k}(y_k)$ satisfy $z = r_k x + y_k$. Using the change of variables $x \mapsto r_k x + y_k$, combined with (5.4.15) and (5.4.20), we find

$$\begin{aligned}\int_{B_1^+(0)} x_{m+1}^\beta \langle \nabla w_k, \nabla\phi\rangle \,\mathrm{d}x &= \frac{r_k}{\tilde{\varepsilon}_k} \int_{B_1^+(0)} x_{m+1}^\beta \langle \nabla\partial_i v_k(r_k x + y_k), \nabla\phi(x)\rangle \,\mathrm{d}x \\ &= \frac{r_k^{-m-\beta}}{\tilde{\varepsilon}_k} \int_{B_{r_k}^+(y_k)} z_{m+1}^\beta \left\langle \nabla\partial_i v_k, \nabla\phi\left(\frac{z-y_k}{r_k}\right)\right\rangle \mathrm{d}z \\ &= \frac{r_k^{1-m-\beta}}{\tilde{\varepsilon}_k} \int_{B_{r_k}^+(y_k)} z_{m+1}^\beta \left\langle \nabla\partial_i v_k, \nabla\tilde{\phi}\right\rangle \mathrm{d}z. \end{aligned} \tag{5.4.21}$$

As $\tilde{\phi} \in C_0^\infty(B_{r_k}(y_k);\mathbb{R}^n)$ and v_k is a minimiser of E^β relative to $\mathcal{O}$, we have

$$\begin{aligned}
&\left|\int_{B^+_{r_k}(y_k)} z_{m+1}^\beta \left\langle \nabla\partial_i v_k, \nabla\tilde{\phi}\right\rangle \mathrm{d}z\right| \\
&= \left|\int_{B^+_{r_k}(y_k)} z_{m+1}^\beta \left\langle \tilde{\phi}, 2A(v_k)(\nabla\partial_i v_k, \nabla v_k) + DA(v_k)(\nabla v_k, \nabla v_k, \partial_i v_k)\right\rangle \mathrm{d}z\right| \\
&\leq ||\tilde{\phi}||_{L^\infty(B^+_{r_k}(y_k);\mathbb{R}^n)} C \int_{B^+_{r_k}(y_k)} z_{m+1}^\beta (|\nabla\partial_i v_k| + 1)\mathrm{d}z. \qquad (5.4.22)
\end{aligned}$$

It follows from (5.4.13) that $\frac{r_k^2}{\tilde{\varepsilon}_k^2} < \frac{1}{\delta}$. Using this fact, changing variables and using (5.4.15) again, we find

$$\begin{aligned}
\int_{B^+_{r_k}(y_k)} z_{m+1}^\beta (|\nabla\partial_i v_k| + 1)\mathrm{d}z &= r_k^{1+m+\beta} \int_{B_1^+(0)} x_{m+1}^\beta (|\nabla\partial_i v_k(r_k x + y_k)| + 1)\mathrm{d}x \\
&= \tilde{\varepsilon}_k r_k^{m+\beta} \int_{B_1^+(0)} x_{m+1}^\beta (|\nabla w_k| + \frac{r_k}{\tilde{\varepsilon}_k})\mathrm{d}x \\
&\leq \tilde{\varepsilon}_k r_k^{m+\beta} \int_{B_1^+(0)} x_{m+1}^\beta (|\nabla w_k| + \delta^{-\frac{1}{2}})\mathrm{d}x. \qquad (5.4.23)
\end{aligned}$$

We combine (5.4.21), (5.4.22) and (5.4.23) with the fact that $||\tilde{\phi}||_{L^\infty(B^+_{r_k}(y_k);\mathbb{R}^n)} = ||\phi||_{L^\infty(B_1^+(0);\mathbb{R}^n)}$ to see that

$$\left|\int_{B_1^+(0)} x_{m+1}^\beta \left\langle \nabla w_k, \nabla\phi\right\rangle \mathrm{d}x\right| \leq C||\phi||_{L^\infty(B_1^+(0);\mathbb{R}^n)} r_k \int_{B_1^+(0)} x_{m+1}^\beta (|\nabla w_k| + \delta^{-\frac{1}{2}})\mathrm{d}x. \qquad (5.4.24)$$

Now note that for any $\phi \in C_0^\infty(B_1(0);\mathbb{R}^n)$, the weak convergence of w_{k_j} to w in $W_\beta^{1,2}(B_1^+(0);\mathbb{R}^n)$, combined with (5.4.24) and (5.4.16) yields

$$\begin{aligned}
\left|\int_{B_1^+(0)} x_{m+1}^\beta \langle\nabla w, \nabla\phi\rangle \mathrm{d}x\right| &= \lim_{j\to\infty} \left|\int_{B_1^+(0)} x_{m+1}^\beta \langle\nabla w_{k_j}, \nabla\phi\rangle \mathrm{d}x\right| \\
&\leq C||\phi||_{L^\infty} \lim_{j\to\infty} r_{k_j} \int_{B_1^+(0)} x_{m+1}^\beta (|\nabla w_{k_j}| + \delta^{-\frac{1}{2}})\mathrm{d}x \\
&= 0
\end{aligned}$$

since $r_{k_j} \to 0$. Hence w is a weak solution of (2.4.5) in $B_1^+(0)$.

We also conclude, by taking limits in (5.4.16), (5.4.17) and (5.4.18), that

$$\int_{B_1^+(0)} x_{m+1}^\beta |\nabla w|^2 \,\mathrm{d}x \leq 1, \qquad (5.4.25)$$

$$\overline{w}_{B_\theta^+(0),\beta} = \frac{1}{\int_{B_\theta^+(0)} x_{m+1}^\beta \mathrm{d}x} \int_{B_\theta^+(0)} x_{m+1}^\beta w \mathrm{d}x = 0 \tag{5.4.26}$$

and

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta \left|w\right|^2 \mathrm{d}x \geq \delta \tag{5.4.27}$$

respectively, where we have used the Rellich Compactness Lemma, Lemma 2.3.5.2, to take the limit in (5.4.25) and (5.4.27). Now, in view of (5.4.26), the Poincaré inequality, Lemma 2.3.3.3, yields

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |w|^2 \mathrm{d}x \leq C\theta^{1-m-\beta} \int_{B_\theta^+(0)} x_{m+1}^\beta |\nabla w|^2 \mathrm{d}x. \tag{5.4.28}$$

Lastly, since w is a weak solution of (2.4.5) we may apply Corollary 2.4.3.2 to w with $\theta \leq \frac{1}{4}$ (so that $2\theta \leq \frac{1}{2}$). This gives a positive constant C (independent of θ) and a $\gamma \in (0,1)$ such that

$$\theta^{1-m-\beta} \int_{B_\theta^+(0)} x_{m+1}^\beta |\nabla w|^2 \mathrm{d}x \leq C(2\theta)^{2\gamma}. \tag{5.4.29}$$

Combining (5.4.28) and (5.4.29) we see that

$$\theta^{-(1+m+\beta)} \int_{B_\theta^+(0)} x_{m+1}^\beta |w|^2 \mathrm{d}x \leq C(2\theta)^{2\gamma}. \tag{5.4.30}$$

This holds for all fixed $\theta \in (0, \frac{1}{4}]$ and we choose $\theta < 2^{-1}\left(\frac{\delta}{C}\right)^{\frac{1}{2\gamma}}$ so that (5.4.30) contradicts (5.4.27). Hence the lemma is proved. □

5.5 Improved Control in the Poincaré Inequality for First Derivatives in the Interior

We need a counterpart to Lemma 5.4.0.1 which holds on a class of balls with closure contained in the interior of $\mathbb{R}_+^{m+1}$.

Lemma 5.5.0.1. *Let $i = 1, \ldots, m$. For every $\delta > 0$ there exist numbers $\varepsilon > 0$, $\tau \in (0,1)$ and $\theta \in (0, \frac{1}{4}]$ such that the following holds. Suppose v is a minimiser of E^β relative to $\mathcal{O}$. Let $B_R^+(x_0) \subset \mathbb{R}_+^{m+1}$ be a half-ball with $R \leq 1$ and $\overline{\partial^0 B_R^+(x_0)} \subset \mathcal{O}$. If*

$$R^{1-m-\beta} \int_{B_R^+(x_0)} x_{m+1}^\beta \left|\nabla v\right|^2 \mathrm{d}x \leq \varepsilon^2,$$

then, for every $B_r(y) \in \mathcal{B}_4(x_0, R, \tau R)$, either

$$r^{1-m} \int_{B_r(y)} |\nabla \partial_i v|^2 \, \mathrm{d}x \le \delta \frac{1}{R^2} \left(\frac{r}{R}\right)^2$$

or

$$(\theta r)^{-(1+m)} \int_{B_{\theta r}(y)} \left|\partial_i v - \overline{\partial_i v}_{B_{\theta r}(y)}\right|^2 \mathrm{d}x \le \delta r^{1-m} \int_{B_r(y)} |\nabla \partial_i v|^2 \, \mathrm{d}x.$$

Proof. The method of proof is similar to the proof of Lemma 5.4.0.1. We observe that the lemma is invariant under scaling and translation with respect to x_0 in $\partial\mathbb{R}^{m+1}_+$ in an analogous way to the proof of Lemma 5.4.0.1. Hence we assume $R = 1$, $x_0 = 0$, v is a minimiser of E^β relative to $\mathcal{O}$ and $\overline{\partial^0 B_1^+(0)} \subset \mathcal{O}$. We argue by contradiction. We will show that if the lemma were false, then we may construct a weak solution of $\operatorname{div}((1 + ax_{m+1})^\beta \nabla w) = 0$ in $B_1(0)$, for a carefully chosen $a \in (0, \frac{1}{4}]$, whose $L^2(B_1(0); \mathbb{R}^n)$ norm is bounded below and strictly above by the same number, a contradiction.

Suppose the statement is false. Then there exists a $\delta > 0$ such that, for any fixed $\theta \in (0, \frac{1}{4}]$ we may find a sequence $(v_k)_{k\in\mathbb{N}}$ of minimising harmonic maps, relative to $\mathcal{O}$, with $v_k \in W^{1,2}_\beta(B_1^+(0); N)$ such that

$$\int_{B_1^+(0)} x_{m+1}^\beta |\nabla v_k|^2 \, \mathrm{d}x := \varepsilon_k^2 \to 0,$$

and, furthermore, a sequence of numbers $0 < \tau_k \to 0$, balls $B_{r_k}(y_k) \in \mathcal{B}_4(0, 1, \tau_k)$, and numbers $0 < r_k \le \tau_k \to 0$ such that

$$r_k^{1-m} \int_{B_{r_k}(y_k)} |\nabla \partial_i v_k|^2 \, \mathrm{d}x > \delta r_k^2 \tag{5.5.1}$$

and

$$\begin{aligned}(\theta r_k)^{-(1+m)} &\int_{B_{\theta r_k}(y_k)} \left|\partial_i v_k - \overline{(\partial_i v_k)}_{B_{\theta r_k}(y_k)}\right|^2 \mathrm{d}x \\ &> \delta r_k^{1-m} \int_{B_{r_k}(y_k)} |\nabla \partial_i v_k|^2 \, \mathrm{d}x. \end{aligned} \tag{5.5.2}$$

We show that $r_k^{1-m} \int_{B_{r_k}(y_k)} |\nabla \partial_i v_k|^2 \, \mathrm{d}x \to 0$. Discarding as many v_k as necessary and re-indexing the resulting sequence, we may suppose that $\varepsilon_k^2 \le \varepsilon$ for every k, where ε is the number from Theorem 3.12.1.1. Furthermore we may choose this sequence such that $2\tau_k \le \frac{\theta_0}{12}$, where θ_0 is the number given by Theorem 3.12.1.1, so that $B_{r_k}(y_k) \subset B_{2r_k}(y_k) \in \mathcal{B}(0, \frac{\theta_0}{6}, \frac{\theta_0}{24})$. It follows that $B_{2r_k}(y_k) \subset B^+_{\frac{\theta_0}{6}}(0)$.

Applying Theorem 5.1.0.1, we see that each v_k satisfies

$$||\nabla v_k||^2_{L^\infty(B^+_{\frac{\theta_0}{3}}(0);\mathbb{R}^{(m+1)n})} \leq \frac{C}{\theta_0^2}\theta_0^{1-m-\beta}\int_{B^+_{\theta_0}(0)} x^\beta_{m+1}|\nabla v_k|^2 \mathrm{d}x.$$

Furthermore, the boundary monotonicity formula, Lemma 3.3.1.1, implies that

$$\theta_0^{1-m-\beta}\int_{B^+_{\theta_0}(0)} x^\beta_{m+1}|\nabla v_k|^2 \mathrm{d}x \leq \int_{B^+_1(0)} x^\beta_{m+1}\left|\nabla v_k\right|^2 \mathrm{d}x = \varepsilon_k^2.$$

Thus,

$$||\nabla v_k||^2_{L^\infty(B^+_{\frac{\theta_0}{3}}(0);\mathbb{R}^{(m+1)n})} \leq \frac{C}{\theta_0^2}\varepsilon_k^2 \to 0. \tag{5.5.3}$$

Now we make use of the Caccioppoli-type inequality, Lemma 5.3.0.1, with $\lambda = 0$, noting that by Lemma 5.2.0.1 we have $\nabla\partial_i v_k \in L^2_\beta(B^+_{\frac{\theta_0}{6}}(0);\mathbb{R}^{(m+1)n})$ for $i = 1,\ldots,m$. We apply the lemma on $B_{2r_k}(y_k) \subset B^+_{\frac{\theta_0}{6}}(0)$. This yields

$$\begin{aligned}
&\int_{B_{r_k}(y_k)} x^\beta_{m+1}\left|\nabla\partial_i v_k\right|^2 \mathrm{d}x \\
&\leq C\left(||\nabla v_k||^2_{L^\infty(B_{2r_k}(y_k);\mathbb{R}^{(m+1)n})} + \frac{1}{r_k^2}\right)\int_{B_{2r_k}(y_k)} x^\beta_{m+1}\left|\partial_i v_k\right|^2 \mathrm{d}x \\
&\quad + C||\nabla v_k||^6_{L^\infty(B_{2r_k}(y_k);\mathbb{R}^{(m+1)n})}\int_{B_{2r_k}(y_k)} x^\beta_{m+1}\mathrm{d}x.
\end{aligned} \tag{5.5.4}$$

Since $||\nabla v_k||_{L^\infty(B_{2r_k}(y_k);\mathbb{R}^{(m+1)n})} \leq ||\nabla v_k||_{L^\infty(B^+_{\frac{\theta_0}{3}}(0);\mathbb{R}^{(m+1)n})}$ we may combine (5.5.3) with (5.5.4), multiplying the latter by $(y_k)^{-\beta}_{m+1}r_k^{1-m}$ and using (3.4.4) in Section 3.4, to see that

$$\begin{aligned}
&r_k^{1-m}\int_{B_{r_k}(y_k)} \left|\nabla\partial_i v_k\right|^2 \mathrm{d}x \\
&\leq Cr_k^2||\nabla v_k||^4_{L^\infty(B^+_{\frac{\theta_0}{3}}(0);\mathbb{R}^{(m+1)n})} + C||\nabla v_k||^2_{L^\infty(B^+_{\frac{\theta_0}{3}}(0);\mathbb{R}^{(m+1)n})} \\
&\quad + Cr_k^2||\nabla v_k||^6_{L^\infty(B^+_{\frac{\theta_0}{3}}(0);\mathbb{R}^{(m+1)n})} \\
&\leq C(r_k^2\varepsilon_k^4 + \varepsilon_k^2 + r_k^2\varepsilon_k^6) \to 0,
\end{aligned} \tag{5.5.5}$$

where C depends on m, N, β and θ_0 and hence only on m, N, β. From (5.5.5), we conclude that

$$r_k^{1-m}\int_{B_{r_k}(y_k)} \left|\nabla\partial_i v_k\right|^2 \mathrm{d}x := \tilde{\varepsilon}_k^2 \to 0. \tag{5.5.6}$$

It follows from (5.5.1) and (5.5.2) that

$$\tilde{\varepsilon}_k^2 > \delta r_k^2 \tag{5.5.7}$$

and

$$(\theta r_k)^{-(1+m)} \int_{B_{\theta r_k}(y_k)} \left| \partial_i v_k - \overline{(\partial_i v_k)}_{B_{\theta r_k}(y_k)} \right|^2 \mathrm{d}x > \delta \tilde{\varepsilon}_k^2. \tag{5.5.8}$$

To find a weak solution of $\operatorname{div}((1 + a x_{m+1})^\beta \nabla w) = 0$ in $B_1(0)$ for some $a \in (0, \frac{1}{2}]$, we consider the normalised sequence

$$w_k(x) = \frac{\partial_i v_k(r_k x + y_k) - \overline{(\partial_i v_k)}_{B_{\theta r_k}(y_k)}}{\tilde{\varepsilon}_k}.$$

We calcuate

$$\nabla w_k(x) = \frac{r_k}{\tilde{\varepsilon}_k} \nabla \partial_i v_k \left(r_k x + y_k \right). \tag{5.5.9}$$

Using the change of variables $x \mapsto r_k x + y_k$ we thus find

$$\int_{B_1(0)} \left| \nabla w_k \right|^2 \mathrm{d}x = 1, \tag{5.5.10}$$

$$\overline{(w_k)}_{B_\theta(0)} = \frac{1}{\int_{B_\theta(0)} \mathrm{d}x} \int_{B_\theta(0)} w_k \mathrm{d}x = 0 \tag{5.5.11}$$

and, also using (5.5.8),

$$\theta^{-(1+m)} \int_{B_\theta(0)} \left| w_k \right|^2 \mathrm{d}x > \delta. \tag{5.5.12}$$

As a result of (5.5.11), we may write $w_k = w_k - \overline{(w_k)}_{B_\theta(0)}$. Hence we can apply the Poincaré inequality, as given in estimate 7.45 of [21], to see that

$$\int_{B_1(0)} |w_k|^2 \mathrm{d}x = \int_{B_1(0)} |w_k - \overline{(w_k)}_{B_\theta(0)}|^2 \mathrm{d}x \leq C \theta^{-(1+m)} \int_{B_1(0)} \left| \nabla w_k \right|^2 \mathrm{d}x. \tag{5.5.13}$$

Together, (5.5.10) and (5.5.13) guarantee that $(w_k)_{k \in \mathbb{N}}$ is a bounded sequence in $W^{1,2}(B_1(0); \mathbb{R}^n)$. The Rellich Compactness lemma, [46] Section 1.3 Lemma 1, thus yields a subsequence $(w_{k_j})_{j \in \mathbb{N}}$ which converges weakly in $W^{1,2}(B_1(0); \mathbb{R}^n)$ and strongly in $L^2(B_1(0); \mathbb{R}^n)$ to some $w \in W^{1,2}(B_1(0); \mathbb{R}^n)$.

In order to derive the required contradiction, we will show that w is a solution of some PDE which satisfies a mean value inequality, similarly to Euclidean harmonic functions. To this end, we examine the sequence of maps $(f_{k_j})_{j\in\mathbb{N}}$ where f_k is defined by

$$f_k(x) = \left(1 + \frac{r_k}{(y_k)_{m+1}} x_{m+1}\right)^\beta$$

for each $k \in \mathbb{N}$. Note that the sequence of numbers $a_{k_j} = \frac{r_{k_j}}{(y_{k_j})_{m+1}}$ satisfies $0 \le a_{k_j} \le \frac{1}{4}$ for every j, since each $B_{r_k}(y_k) \in \mathcal{B}_4(0, 1, \tau_k)$, and thus there is a subsequence, which we also index with k_j which converges to a number a with $0 \le a \le \frac{1}{4}$ as $j \to \infty$. Furthermore, the sequence f_{k_j} is uniformly bounded and equicontinuous so, by the Arzelà-Ascoli theorem, there is a uniformly convergent subsequence which we again index by k_j. Since $f_{k_j}(x)$ converges pointwise to $f(x) = (1 + ax_{m+1})^\beta$ this must also be the uniform limit of f_{k_j}.

We will now show that w is a weak solution of $\operatorname{div}((1 + ax_{m+1})^\beta \nabla w) = 0$ in $B_1(0)$. Let $\phi \in C_0^\infty(B_1(0); \mathbb{R}^n)$ and define $\tilde{\phi} \in C_0^\infty(B_{r_k}(y_k); \mathbb{R}^n)$ by $\tilde{\phi}(z) = \phi\left(\frac{z-y_k}{r_k}\right)$. We observe that

$$\nabla\tilde{\phi}(z) = \nabla\left(\phi\left(\frac{z-y_k}{r_k}\right)\right) = \frac{1}{r_k}\nabla\phi\left(\frac{z-y_k}{r_k}\right) = \frac{1}{r_k}\nabla\phi(x) \tag{5.5.14}$$

where $x \in B_1(0)$ and $z \in B_{r_k}(y_k)$ satisfy $z = r_k x + y_k$. Using the change of variables $x \mapsto r_k x + y_k$, combined with (5.5.9) and (5.5.14), we find

$$\begin{aligned}
&\int_{B_1(0)} f_k(x) \langle \nabla w_k, \nabla\phi\rangle \,\mathrm{d}x \\
&= \frac{r_k}{\tilde{\varepsilon}_k} \int_{B_1(0)} f_k(x) \langle \nabla\partial_i v_k(r_k x + y_k), \nabla\phi(x)\rangle \,\mathrm{d}x \\
&= (y_k)_{m+1}^{-\beta} \frac{r_k^{-m}}{\tilde{\varepsilon}_k} \int_{B_{r_k}(y_k)} z_{m+1}^\beta \left\langle \nabla\partial_i v_k, \nabla\phi\left(\frac{z-y_k}{r_k}\right)\right\rangle \mathrm{d}z \\
&= (y_k)_{m+1}^{-\beta} \frac{r_k^{1-m}}{\tilde{\varepsilon}_k} \int_{B_{r_k}(y_k)} z_{m+1}^\beta \left\langle \nabla\partial_i v_k, \nabla\tilde{\phi}\right\rangle \mathrm{d}z.
\end{aligned} \tag{5.5.15}$$

As $\tilde{\phi} \in C_0^\infty(B_{r_k}(y_k); \mathbb{R}^n)$ and v_k is a minimiser we have

$$\begin{aligned}
&\left|\int_{B_{r_k}(y_k)} z_{m+1}^\beta \left\langle \nabla\partial_i v_k, \nabla\tilde{\phi}\right\rangle \mathrm{d}z\right| \\
&= \left|\int_{B_{r_k}(y_k)} z_{m+1}^\beta \left\langle \tilde{\phi}, 2A(v_k)(\nabla\partial_i v_k, \nabla v_k) + DA(v_k)(\nabla v_k, \nabla v_k, \partial_i v_k)\right\rangle \mathrm{d}z\right| \\
&\le \|\tilde{\phi}\|_{L^\infty(B_{r_k}(y_k);\mathbb{R}^n)} C \int_{B_{r_k}(y_k)} z_{m+1}^\beta (|\nabla\partial_i v_k| + 1)\mathrm{d}z.
\end{aligned} \tag{5.5.16}$$

It follows from (5.5.7) that $\frac{r_k^2}{\tilde{\varepsilon}_k^2} \leq \delta^{-1}$. Using this fact, changing variables and using (5.5.9) again, we find

$$\begin{aligned}
&\int_{B_{r_k}(y_k)} z_{m+1}^{\beta}(|\nabla \partial_i v_k| + 1)\mathrm{d}z \\
&= (y_k)_{m+1}^{\beta} r_k^{1+m} \int_{B_1(0)} f_k(x)(|\nabla \partial_i v_k(r_k x + y_k)| + 1)\mathrm{d}x \\
&= (y_k)_{m+1}^{\beta} \tilde{\varepsilon}_k r_k^m \int_{B_1(0)} f_k(x)(|\nabla w_k| + \frac{r_k}{\tilde{\varepsilon}_k})\mathrm{d}x \\
&\leq (y_k)_{m+1}^{\beta} \tilde{\varepsilon}_k r_k^m \int_{B_1(0)} f_k(x)(|\nabla w_k| + \delta^{-\frac{1}{2}})\mathrm{d}x. \qquad (5.5.17)
\end{aligned}$$

We combine (5.5.15), (5.5.16) and (5.5.17) with the fact that $||\tilde{\phi}||_{L^\infty(B_{r_k}(y_k);\mathbb{R}^n)} = ||\phi||_{L^\infty(B_1(0);\mathbb{R}^n)}$ and $\sup_{B_1(0)} |f_k|$ is uniformly bounded in k to see that

$$\left| \int_{B_1(0)} f_k(x) \langle \nabla w_k, \nabla \phi \rangle \, \mathrm{d}x \right| \leq C ||\phi||_{L^\infty(B_1(0);\mathbb{R}^n)} r_k \int_{B_1(0)} (|\nabla w_k| + \delta^{-\frac{1}{2}})\mathrm{d}x. \qquad (5.5.18)$$

Next we show that

$$\int_{B_1(0)} f(x)\langle \nabla w, \nabla \phi \rangle \mathrm{d}x = \lim_{j\to\infty} \int_{B_1(0)} f_{k_j}(x)\langle \nabla w_{k_j}, \nabla \phi \rangle \mathrm{d}x.$$

To see this, note that

$$\begin{aligned}
&\int_{B_1(0)} f(x)\langle \nabla w, \nabla \phi \rangle \mathrm{d}x - \int_{B_1(0)} f_{k_j}(x)\langle \nabla w_{k_j}, \nabla \phi \rangle \mathrm{d}x \\
&= \int_{B_1(0)} f(x)\langle \nabla w - \nabla w_{k_j}, \nabla \phi \rangle \mathrm{d}x + \int_{B_1(0)} (f(x) - f_{k_j}(x))\langle \nabla w_{k_j}, \nabla \phi \rangle \mathrm{d}x.
\end{aligned}$$

The weak convergence of w_{k_j} to w in $W^{1,2}(B_1(0);\mathbb{R}^n)$ guarantees that

$$\lim_{j\to\infty} \int_{B_1(0)} f(x)\langle \nabla w - \nabla w_{k_j}, \nabla \phi \rangle \mathrm{d}x = 0.$$

Furthermore, we apply Hölder's inequality to see that

$$\begin{aligned}
&\left| \int_{B_1(0)} (f(x) - f_{k_j}(x))\langle \nabla w_{k_j}, \nabla \phi \rangle \mathrm{d}x \right| \\
&\leq C ||\nabla \phi||_{L^\infty(B_1(0);\mathbb{R}^{(m+1)n})} \sup_{B_1(0)} |f - f_{k_j}| \left(\int_{B_1(0)} |\nabla w_{k_j}|^2 \mathrm{d}x \right)^{\frac{1}{2}}.
\end{aligned}$$

Since ∇w_{k_j} is bounded in L^2 by (5.5.10) we have

$$\left|\int_{B_1(0)} (f(x) - f_{k_j}(x))\langle \nabla w_{k_j}, \nabla\phi\rangle \mathrm{d}x\right| \le C||\nabla\phi||_{L^\infty(B_1(0);\mathbb{R}^{(m+1)n})} \sup_{B_1(0)} |f - f_{k_j}|$$
$$\to 0 \text{ as } j \to \infty,$$

since $f_{k_j} \to f$ uniformly. Thus

$$\int_{B_1(0)} f(x)\langle \nabla w, \nabla\phi\rangle \mathrm{d}x = \lim_{j\to\infty} \int_{B_1(0)} f_{k_j}(x)\langle \nabla w_{k_j}, \nabla\phi\rangle \mathrm{d}x. \tag{5.5.19}$$

Using (5.5.10), (5.5.18) and (5.5.19) we conclude that for every $\phi \in C_0^\infty(B_1(0);\mathbb{R}^n)$ we have

$$\begin{aligned}\left|\int_{B_1(0)} f(x)\langle \nabla w, \nabla\phi\rangle \mathrm{d}x\right| &= \lim_{j\to\infty}\left|\int_{B_1(0)} f_{k_j}(x)\langle \nabla w_{k_j}, \nabla\phi\rangle \mathrm{d}x\right| \\ &\le C||\phi||_{L^\infty(B_1(0);\mathbb{R}^n)} \lim_{j\to\infty} r_{k_j} \int_{B_1(0)} \left(|\nabla w_{k_j}| + \delta^{-\frac{1}{2}}\right) \mathrm{d}x \\ &= 0.\end{aligned}$$

Hence w is a weak solution of $\operatorname{div}((1 + ax_{m+1})^\beta \nabla w) = 0$ in $B_1(0)$. By linear elliptic regularity theory, w is smooth in $B_1(0)$. We also conclude by taking limits in (5.5.10), (5.5.11) and (5.5.12) that

$$\int_{B_1(0)} |\nabla w|^2 \mathrm{d}x \le 1, \tag{5.5.20}$$

$$\overline{w}_{B_\theta(0)} = \frac{1}{\int_{B_\theta(0)} \mathrm{d}x} \int_{B_\theta(0)} w \mathrm{d}x = 0 \tag{5.5.21}$$

and

$$\theta^{-(1+m)} \int_{B_\theta(0)} |w|^2 \mathrm{d}x \ge \delta \tag{5.5.22}$$

respectively using the Rellich compactness lemma for (5.5.20) and (5.5.22). Now, in view of (5.5.21) the Poincaré inequality yields

$$\theta^{-(1+m)} \int_{B_\theta(0)} |w|^2 \mathrm{d}x \le C\theta^{1-m} \int_{B_\theta(0)} |\nabla w|^2 \mathrm{d}x. \tag{5.5.23}$$

We recall that ∇w satisfies a mean value inequality, namely $\sup_{B_\theta(0)} |\nabla w|^2 \le$

$C(m,\beta)\int_{B_1(0)} |\nabla w|^2 \mathrm{d}x$ as shown in theorem 2.1 in section III of [20]. Hence

$$\theta^{1-m}\int_{B_\theta(0)} |\nabla w|^2 \mathrm{d}x \leq C\theta^2 \int_{B_1(0)} |\nabla w|^2 \mathrm{d}x. \tag{5.5.24}$$

Combining (5.5.20) with (5.5.23) and (5.5.24) we see that

$$\theta^{-(1+m)}\int_{B_\theta(0)} |w|^2 \mathrm{d}x \leq C\theta^2. \tag{5.5.25}$$

This holds for all fixed $\theta \in (0, \frac{1}{2}]$ and we choose $\theta < \left(\frac{\delta}{C}\right)^{\frac{1}{2}}$ then (5.5.25) contradicts (5.5.22). Hence the lemma is proved. $\square$

5.6 Hölder Continuity of First Order Derivatives

The culmination of all of our theory so far is the following ε-regularity theorem for minimisers of E^β relative to $\mathcal{O}$.

Theorem 5.6.0.1. *Let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ be a minimiser of E^β relative to $\mathcal{O}$, let $\varepsilon > 0$ be the number from Theorem 3.12.1.1 and let $B^+_R(x_0)$ be a half-ball with $R \leq 1$ and $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$. Suppose $R^{1-m-\beta}\int_{B^+_R(x_0)} x^\beta_{m+1}|\nabla v|^2 \mathrm{d}x \leq \varepsilon$. Then there is a $\theta = \theta(m,N,\beta) \in (0,1)$ and a $\gamma = \gamma(m,N,\beta) \in (0,1)$ such that $\partial_i v \in C^{0,\gamma}(\overline{B^+_{\theta R}(x_0)}; \mathbb{R}^n)$ for $i = 1,\ldots,m$.*

Proof. First we observe that the statement of the lemma is invariant under rescaling and translation by any point in $\partial\mathbb{R}^{m+1}_+$. If the lemma is true on $B^+_1(0)$ for minimisers of E^β relative to $\tilde{\mathcal{O}}$ whenever $\overline{\partial^0 B^+_1(0)} \subset \tilde{\mathcal{O}}$, then we may obtain the lemma on $B^+_R(x_0)$ for a minimiser of E^β relative to $\mathcal{O}$ whenever $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}$ and $R \leq 1$. This follows from rescaling using the map $x \mapsto Rx + x_0$, defined for $x \in B^+_1(0)$, and applying the lemma on $B^+_1(0)$ to the map $v_R(x) = v(Rx + x_0)$. Thus we will assume $R = 1$ and $\overline{\partial^0 B^+_1(0)} \subset \mathcal{O}$.

First we observe that the combination of Theorem 3.12.1.1, Lemma 5.1.0.1 and Lemma 5.2.0.1 yield a $\tilde{\theta} = \tilde{\theta}(m,N,\beta) \leq \frac{1}{2}$ and a $\tilde{\gamma} \in (0,1)$ such that $v \in C^{0,\tilde{\gamma}}(\overline{B^+_{\tilde{\theta}}(0)}; N)$ with $\nabla v \in L^\infty(B^+_{\tilde{\theta}}(0); \mathbb{R}^{(m+1)n})$, $\partial_i v \in W^{1,2}_\beta(B^+_{\tilde{\theta}}(0); \mathbb{R}^n)$ for $i = 1,\ldots,m$. Applying Lemma 5.3.0.1, re-formulating the statement slightly, we see that

$$
\begin{aligned}
&\int_{B_{\frac{\rho}{2}}(y)\cap\mathbb{R}^{m+1}_{+}} x_{m+1}^{\beta}\left|\nabla\partial_i v\right|^2 \mathrm{d}x \\
&\leq \frac{C}{\rho^2}\int_{B_{\rho}(y)\cap\mathbb{R}^{m+1}_{+}} x_{m+1}^{\beta}\left|\partial_i v-\lambda\right|^2 \mathrm{d}x \\
&\quad + C|\lambda|^2||\nabla v||^2_{L^{\infty}(B_{\rho}(y)\cap\mathbb{R}^{m+1}_{+};\mathbb{R}^{(m+1)n})}\int_{B_{\rho}(y)\cap\mathbb{R}^{m+1}_{+}} x_{m+1}^{\beta}\mathrm{d}x \\
&\quad + C||\nabla v||^4_{L^{\infty}(B_{\rho}(y)\cap\mathbb{R}^{m+1}_{+};\mathbb{R}^{(m+1)n})}\int_{B_{\rho}(y)\cap\mathbb{R}^{m+1}_{+}} x_{m+1}^{\beta}\mathrm{d}x \\
&\quad + C||\nabla v||^6_{L^{\infty}(B_{\rho}(y)\cap\mathbb{R}^{m+1}_{+};\mathbb{R}^{(m+1)n})}\int_{B_{\rho}(y)\cap\mathbb{R}^{m+1}_{+}} x_{m+1}^{\beta}\mathrm{d}x \qquad (5.6.1)
\end{aligned}
$$

for any $B_{\rho}(y)\subset B_{\tilde{\theta}}(0)$ with $y_{m+1}\geq 0$. We apply Lemmata 5.4.0.1 and 5.5.0.1 with $\delta=\frac{1}{2}\frac{1}{2^{m+2}C}$ where C is the constant from (5.6.1). The powers of 2 appear in the choice of δ as we will later need to multiply (5.6.1) by 2^{m+1} or $2^{m+1+\beta}$, depending on the context, and the given δ ensures the factor outside the integral involving $\partial_i v-\lambda$ remains less than a half. An application of the lemmata gives numbers $\varepsilon_1>0$, $\tau_1\in(0,1)$ and $\theta_1\in(0,\frac{1}{4}]$ and $\varepsilon_2>0$, $\tau_2\in(0,1)$ and $\theta_2\in(0,\frac{1}{4}]$ respectively such that the following holds. Let $R_1,R_2\in(0,1]$. If

$$
R_1^{1-m-\beta}\int_{B^{+}_{R_1}(0)} x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x\leq\varepsilon_1^2 \qquad (5.6.2)
$$

then for every $B^{+}_{r_1}(y)\in\mathcal{B}^{+}(0,R_1,\tau_1R_1)$ either

$$
r_1^{1-m-\beta}\int_{B^{+}_{r_1}(y)} x_{m+1}^{\beta}\left|\nabla\partial_i v\right|^2\mathrm{d}x\leq\delta\frac{1}{R_1^2}\left(\frac{r_1}{R_1}\right)^2 \qquad (5.6.3)
$$

or

$$
\begin{aligned}
&(\theta_1r_1)^{-(1+m+\beta)}\int_{B^{+}_{\theta_1r_1}(y)} x_{m+1}^{\beta}\left|\partial_i v-\overline{\partial_i v}_{B^{+}_{\theta_1r_1}(y),\beta}\right|^2\mathrm{d}x \\
&\leq\delta {r_1}^{1-m-\beta}\int_{B^{+}_{r_1}(y)} x_{m+1}^{\beta}\left|\nabla\partial_i v\right|^2\mathrm{d}x. \qquad (5.6.4)
\end{aligned}
$$

Furthermore, if

$$
R_2^{1-m-\beta}\int_{B^{+}_{R_2}(0)} x_{m+1}^{\beta}|\nabla v|^2\mathrm{d}x\leq\varepsilon_2^2 \qquad (5.6.5)
$$

then for every $B_{r_2}(y) \in \mathcal{B}_4(0, R_2, \tau_2 R_2)$ either

$$r_2{}^{1-m} \int_{B_{r_2}(y)} |\nabla \partial_i v|^2 \, \mathrm{d}x \leq \delta \frac{1}{R_2^2} \left(\frac{r_2}{R_2} \right)^2 \tag{5.6.6}$$

or

$$(\theta_2 r_2)^{-(1+m)} \int_{B_{\theta_2 r_2}(y)} \left| \partial_i v - \overline{\partial_i v}_{B_{\theta_2 r_2}(y)} \right|^2 \mathrm{d}x \leq \delta r_2{}^{1-m} \int_{B_{r_2}(y)} |\nabla \partial_i v|^2 \, \mathrm{d}x. \tag{5.6.7}$$

We now show that (5.6.2) and (5.6.5) hold simultaneously for some $\tilde{R} = R_1 = R_2$. It follows from the proof of Theorem 3.12.1.1, bearing in mind $R = 1$, that for every $B_r^+(y) \in \mathcal{B}^+(0, 1, \frac{1}{2})$ we have

$$r^{1-m-\beta} \int_{B_r^+(y)} x_{m+1}^\beta |\nabla v|^2 \mathrm{d}x \leq C r^{\tilde{\gamma}}$$

for some $\tilde{\gamma} \in (0, 1)$ and some constant C. In particular, this holds for $y = 0$ and $r \leq \frac{1}{2}$. Hence if $\tilde{R} = \tilde{R}(m, N, \beta) = (\min\{\frac{\varepsilon_1^2}{C}, \frac{\varepsilon_2^2}{C}, \left(\frac{\tilde{\theta}}{2}\right)^{\tilde{\gamma}}\})^{\frac{1}{\tilde{\gamma}}}$ then (5.6.2) and (5.6.5) hold on $B_{\tilde{R}}^+(0)$, that is, with $R_1 = R_2 = \tilde{R}$. We have assumed $\tilde{R} \leq \frac{\tilde{\theta}}{2}$ so that we may later apply (5.6.1) with impunity on any ball or half-ball in $B_{\tilde{R}}^+(0)$. Consequently, either (5.6.3) or (5.6.4) and either (5.6.6) or (5.6.7) hold for some $\tau_1, \theta_1, \tau_2, \theta_2$ depending on δ and thus only on m, N, β. We claim this is sufficient for us to deduce the hypothesis of Lemma 3.5.0.2 hold.

First we show that (3.5.3) holds on every $B_{r_1}^+(y) \in \mathcal{B}^+(0, \tilde{R}, \tau_1 \tilde{R})$. To see this we use an iterative procedure. Consider $r_1{}^{1-m-\beta} \int_{B_{r_1}^+(y)} x_{m+1}^\beta |\nabla \partial_i v|^2 \mathrm{d}x$. We know that (5.6.3) or (5.6.4) holds on $B_{r_1}^+(y)$ with $R_1 = \tilde{R}$. We apply 5.6.1 with $\lambda = \overline{\partial_i v}_{B_{\theta_1 r_1}^+(y), \beta}$, noting that $|\lambda| \leq ||\nabla v||_{L^\infty(B_{\theta_1 r_1}^+(y); \mathbb{R}^{(m+1)n})}$. We deduce that

$$\begin{aligned}
&\left(\frac{\theta_1 r_1}{2} \right)^{1-m-\beta} \int_{B_{\frac{\theta_1 r_1}{2}}^+(y)} x_{m+1}^\beta \left| \nabla \partial_i v \right|^2 \mathrm{d}x \\
&\leq C 2^{m+\beta-1} (\theta_1 r_1)^{-(1+m+\beta)} \int_{B_{\theta_1 r_1}^+(y)} x_{m+1}^\beta \left| \partial_i v - \overline{\partial_i v}_{B_{\theta_1 r_1}^+(y), \beta} \right|^2 \mathrm{d}x \\
&\qquad + C(||\nabla v||_{L^\infty(B_{\tilde{\theta}}^+(0); \mathbb{R}^{(m+1)n})}^4 + ||\nabla v||_{L^\infty(B_{\tilde{\theta}}^+(0); \mathbb{R}^{(m+1)n})}^6) r_1^2.
\end{aligned} \tag{5.6.8}$$

Hence, regardless of which of (5.6.3) or (5.6.4) holds (bearing in mind our choice of δ above), we have

$$
\begin{aligned}
&(\sigma_1 r_1)^{1-m-\beta} \int_{B^+_{\sigma_1 r_1}(y)} x_{m+1}^{\beta} \left|\nabla \partial_i v\right|^2 \mathrm{d}x \\
&\leq \frac{1}{2} {r_1}^{1-m-\beta} \int_{B^+_{r_1}(y)} x_{m+1}^{\beta} |\nabla \partial_i v|^2 \mathrm{d}x + C r_1^2,
\end{aligned} \tag{5.6.9}
$$

where $\sigma_1 = \frac{\theta_1}{2}$ and C may depend on positive powers of $||\nabla v||_{L^\infty(B^+_{\tilde{\theta}}(0);\mathbb{R}^n)}$ and, moreover, on $\tilde{R}, \theta_1, m, N$ and β and hence only on m, N, β as $\tilde{R} = \tilde{R}(m, N, \beta)$ and $\theta_1 = \theta_1(m, N, \beta)$. This holds for any $B^+_{r_1}(y) \in \mathcal{B}^+(0, \tilde{R}, \tau_1 \tilde{R})$. In particular, we may apply (5.6.9) with r_1 replaced by $\sigma_1^k r_1$ for every $k \in \mathbb{N}$. This gives

$$
\begin{aligned}
&\left(\sigma_1^k r_1\right)^{1-m-\beta} \int_{B^+_{\sigma_1^k r_1}(y)} x_{m+1}^{\beta} \left|\nabla \partial_i v\right|^2 \mathrm{d}x \\
&\leq \frac{1}{2} (\sigma_1^{k-1} r_1)^{1-m-\beta} \int_{B^+_{\sigma_1^{k-1} r_1}(y)} x_{m+1}^{\beta} |\nabla \partial_i v|^2 \mathrm{d}x + C(\sigma_1^{k-1} r_1)^2.
\end{aligned} \tag{5.6.10}
$$

We iteratively deduce that

$$
\begin{aligned}
&\left(\sigma_1^k r_1\right)^{1-m-\beta} \int_{B^+_{\sigma_1^k r_1}(y)} x_{m+1}^{\beta} \left|\nabla \partial_i v\right|^2 \mathrm{d}x \\
&\leq \frac{1}{2^k} {r_1}^{1-m-\beta} \int_{B^+_{r_1}(y)} x_{m+1}^{\beta} |\nabla \partial_i v|^2 \mathrm{d}x + C \sum_{j=0}^{k-1} 2^{-j} (\sigma_1^{k-1-j} r_1)^2.
\end{aligned} \tag{5.6.11}
$$

Consider the sum on the right hand side above. Recall that $\sigma_1 = \frac{\theta_1}{2} \in (0, \frac{1}{8}]$. Hence,

$$
\begin{aligned}
C \sum_{j=0}^{k-1} 2^{-j} (\sigma_1^{k-1-j} r_1)^2 &= \frac{C r_1^2}{2^k} \sum_{j=0}^{k-1} 2^{k-j} \sigma_1^{k-1-j} \sigma_1^{k-1-j} \\
&\leq \frac{C r_1^2}{2^k} \sigma_1^{-2} \sum_{j=0}^{k-1} 2^{k-j} \sigma_1^{k-j} \sigma_1^{k-j} \\
&\leq \frac{C r_1^2}{2^k} \sum_{j=0}^{\infty} \sigma_1^{j} \\
&\leq \frac{C r_1^2}{2^k}.
\end{aligned} \tag{5.6.12}
$$

Substituting (5.6.12) into (5.6.11) yields

$$\begin{aligned}&\left(\sigma_1^k r_1\right)^{1-m-\beta}\int_{B^+_{\sigma_1^k r_1}(y)} x_{m+1}^{\beta}\left|\nabla\partial_i v\right|^2\mathrm{d}x\\&\leq \frac{1}{2^k}\left({r_1}^{1-m-\beta}\int_{B^+_{r_1}(y)} x_{m+1}^{\beta}|\nabla\partial_i v|^2\mathrm{d}x + Cr_1^2\right).\end{aligned} \tag{5.6.13}$$

Let $\gamma_1 = -\frac{\ln 2}{\ln\sigma_1}\in(0,1)$. Then $(\sigma_1^k)^{\gamma_1}=2^{-k}$ and so

$$\begin{aligned}&\left(\sigma_1^k r_1\right)^{1-m-\beta}\int_{B^+_{\sigma_1^k r_1}(y)} x_{m+1}^{\beta}\left|\nabla\partial_i v\right|^2\mathrm{d}x\\&\leq \frac{(\sigma_1^k r_1)^{\gamma_1}}{r_1^{\gamma_1}}\left({r_1}^{1-m-\beta}\int_{B^+_{r_1}(y)} x_{m+1}^{\beta}|\nabla\partial_i v|^2\mathrm{d}x + Cr_1^2\right).\end{aligned}$$

Now note that for any $r\leq r_1$ we have $r\in[\sigma_1^{k+1}r_1,\sigma_1^k r_1]$ for some $k\in\mathbb{N}_0$. Thus there is a number $c_1\in[1,\sigma_1^{-1}]$ so that $c_1 r=\sigma_1^k r_1$. Hence we deduce that

$$\begin{aligned}&r^{1-m-\beta}\int_{B^+_{r}(y)} x_{m+1}^{\beta}\left|\nabla\partial_i v\right|^2\mathrm{d}x\\&\leq \sigma_1^{1-m-\beta-\gamma_1}\left(\frac{r}{r_1}\right)^{\gamma_1}\left({r_1}^{1-m-\beta}\int_{B^+_{r_1}(y)} x_{m+1}^{\beta}|\nabla\partial_i v|^2\mathrm{d}x + Cr_1^2\right)\end{aligned} \tag{5.6.14}$$

for any $r\leq r_1$. This holds for any $B^+_{r_1}(y)\in\mathcal{B}^+(0,\tilde{R},\tau_1\tilde{R})$.

We want a similar estimate for ${r_2}^{1-m}\int_{B_{r_2}(y)}\left|\nabla\partial_i v\right|^2\mathrm{d}x$ on balls $B_{r_2}(y)\in\mathcal{B}_4(0,\tilde{R},\tau_2\tilde{R})$. A similar argument which lead to (5.6.14) yields the existence of a $\gamma_2=\gamma_2(m,N,\beta)\in(0,1)$ such that for any $B_{r_2}(y)\in\mathcal{B}_4(0,\tilde{R},\tau_2\tilde{R})$ and any $r\leq r_2$ we have

$$r^{1-m}\int_{B_r(y)}\left|\nabla\partial_i v\right|^2\mathrm{d}x\leq\sigma_2^{1-m-\gamma_2}\left(\frac{r}{r_2}\right)^{\gamma_2}\left({r_2}^{1-m}\int_{B_{r_2}(y)}|\nabla\partial_i v|^2\mathrm{d}x+Cr_2^2\right), \tag{5.6.15}$$

where $\sigma_2=\frac{\theta_2}{2}$. Together, (5.6.14) and (5.6.15) essentially constitute (3.5.3) and (3.5.4) from Lemma 3.5.0.2. We now show that these hypothesis are actually satisfied.

Let $\tau=\min\{\frac{2\tau_1}{3},\tau_2,\frac{1}{2}\}$, $\gamma=\min\{\gamma_1,\gamma_2\}$. We apply (5.6.14) with $r_1=\tau\tilde{R}$.

Then for every $B_r^+(y) \in \mathcal{B}^+(0,\tilde{R},\tau\tilde{R})$ we have $B_{\tau\tilde{R}}^+(y) \in \mathcal{B}^+(0,\tilde{R},\tau_1\tilde{R})$ and hence

$$\begin{aligned} & r^{1-m-\beta}\int_{B_r^+(y)} x_{m+1}^\beta \left|\nabla\partial_i v\right|^2 \mathrm{d}x \\ & \leq C\left(\frac{r}{\tau\tilde{R}}\right)^\gamma \left((\tau\tilde{R})^{1-m-\beta}\int_{B_{\tau\tilde{R}}^+(y)} x_{m+1}^\beta |\nabla\partial_i v|^2 \mathrm{d}x + C(\tau\tilde{R})^2\right). \end{aligned} \tag{5.6.16}$$

Furthermore, applying (5.6.1) with $\lambda = 0$ implies that

$$r^{1-m-\beta}\int_{B_r^+(y)} x_{m+1}^\beta \left|\nabla\partial_i v\right|^2 \mathrm{d}x \leq C(1+(\tau\tilde{R})^2) \leq C, \tag{5.6.17}$$

for every $r \leq \tau\tilde{R}$, where C may depend on positive powers of $||\nabla v||_{L^\infty(B_{\tilde{\theta}}^+(0);\mathbb{R}^{(m+1)n})}$ and m, N, β. We combine (5.6.16) and (5.6.17) to see that for every $B_r^+(y) \in \mathcal{B}^+(0,\tilde{R},\tau\tilde{R})$ we have

$$r^{1-m-\beta}\int_{B_r^+(y)} x_{m+1}^\beta \left|\nabla\partial_i v\right|^2 \mathrm{d}x \leq C\left(\frac{r}{\tau\tilde{R}}\right)^\gamma \leq C_1 r^\gamma, \tag{5.6.18}$$

where C_1 depends on $m, N, \beta, \tilde{R}$ and τ and hence only on m, N and β, which is (3.5.3) in Lemma 3.5.0.2. To see that (3.5.4) holds, we proceed as follows. Let $B_r(y) \in \mathcal{B}_4(0,\tilde{R},\frac{2\tau}{3}\tilde{R})$. Then

$$B_r(y) \subset B_{\frac{y_{m+1}}{4}}(y) \subset B^+_{\frac{3y_{m+1}}{2}}(y^+) \subset B^+_{\tau\tilde{R}}(y^+) \in \mathcal{B}^+(0,\tilde{R},\tau\tilde{R}). \tag{5.6.19}$$

Recalling (3.4.4) from Section 3.4, we note that

$$\begin{aligned} & \left(\frac{y_{m+1}}{4}\right)^{1-m}\int_{B_{\frac{y_{m+1}}{4}}(y)} \left|\nabla\partial_i v\right|^2 \mathrm{d}x \\ & \leq C\left(\frac{3y_{m+1}}{2}\right)^{1-m-\beta}\int_{B^+_{\frac{3y_{m+1}}{2}}(y^+)} x_{m+1}^\beta \left|\nabla\partial_i v\right|^2 \mathrm{d}x. \end{aligned} \tag{5.6.20}$$

Since $y_{m+1} \leq \tau\tilde{R} \leq 1$, applying (5.6.15) on $B_{\frac{y_{m+1}}{4}}(y) \in \mathcal{B}_4(0,\tilde{R},\tau_2\tilde{R})$, using

(5.6.20), and then applying (5.6.18) gives

$$
\begin{aligned}
&r^{1-m}\int_{B_r(y)}|\nabla\partial_i v|^2\,\mathrm{d}x\\
&\leq C\left(\frac{4r}{y_{m+1}}\right)^{\gamma}\left(\left(\frac{y_{m+1}}{4}\right)^{1-m}\int_{B_{\frac{y_{m+1}}{4}}(y)}|\nabla\partial_i v|^2\,\mathrm{d}x+Cy_{m+1}^2\right)\\
&\leq C\left(\frac{r}{y_{m+1}}\right)^{\gamma}\left(C\left(\frac{3y_{m+1}}{2}\right)^{1-m-\beta}\int_{B^+_{\frac{3y_{m+1}}{2}}(y^+)}x_{m+1}^{\beta}\,|\nabla\partial_i v|^2\,\mathrm{d}x+Cy_{m+1}^2\right)\\
&\leq C\left(\frac{r}{y_{m+1}}\right)^{\gamma}\left(\frac{y_{m+1}}{\tau\tilde{R}}\right)^{\gamma}+Cr^{\gamma}y_{m+1}^{2-\gamma}\\
&\leq C_2 r^{\gamma} \qquad\qquad (5.6.21)
\end{aligned}
$$

where C_2 depends on $m, N, \beta, \tilde{R}$ and τ and hence only on m, N and β, which is (3.5.4). Choosing $C = \max\{C_1, C_2\}$ to be the largest constant from (5.6.18) and (5.6.21) we deduce that both hypothesis of Lemma 3.5.0.2 hold simlutaneously for $B_r^+(y) \in \mathcal{B}^+(0, \tilde{R}, \frac{2\tau}{3}\tilde{R})$ and $B_r(y) \in \mathcal{B}_4(0, \tilde{R}, \frac{2\tau}{3}\tilde{R})$. Applying this lemma concludes the proof. □

Theorem 5.6.0.1 yields an improvement to the partial regularity theorem for minimisers of E^β relative to $\mathcal{O}$, Theorem 3.12.2.1, stated in Section 3.12.2. We have the following.

Lemma 5.6.0.1. *Suppose $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ is a minimiser of E^β relative to $\mathcal{O}$. Let $\Sigma_{int} \subset \mathbb{R}^{m+1}_+$ denote the set of points, discussed in Section 3.2, which is relatively closed in $\mathbb{R}^{m+1}_+$ and has Hausdorff dimension $m-2$, such that v is smooth in $\mathbb{R}^{m+1}_+\backslash\Sigma_{int}$. Then there exist $\gamma \in (0,1)$ and a relatively closed set $\Sigma_{bdry} \subset \mathcal{O}$ of vanishing $m+\beta-1$-dimensional Hausdorff measure, with respect to the Euclidean metric, such that $v \in C^{0,\gamma}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma; N)$ and $\partial_i v \in C^{0,\gamma}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma; \mathbb{R}^n)$ for $i = 1, \dots, m$, where $\Sigma = \Sigma_{int} \cup \Sigma_{bdry}$. Furthermore Σ is relatively closed in $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$ and $\mathcal{H}^{m+\beta-1}(\Sigma) = 0$.*

Proof. As in the proof of Lemma 3.12.2.1, define

$$\Sigma_{\mathrm{bdry}} = \{y \in \mathcal{O} : \Theta_v^\beta(y) \geq \varepsilon\}$$

where ε is the number given by Theorem 3.12.1.1 and

$$\Theta_v^\beta(y) = \lim_{r\to 0^+} r^{1-m-\beta}\int_{B_r^+(y)} x_{m+1}^\beta|\nabla v|^2\mathrm{d}x$$

is the density function defined in Section 3.3.1. Let $\Sigma = \Sigma_{\text{int}} \cup \Sigma_{\text{bdry}}$. We have already proved every claim of this lemma for this choice of Σ, with the exception of $\partial_i v \in C^{0,\gamma}((\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma; \mathbb{R}^n)$ for $i = 1, \ldots, m$, in the proof of Lemma 3.12.2.1. We proceed to prove the remaining claim.

Let $x_0 \in (\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$. If $x_0 \in \mathbb{R}^{m+1}_+$ then $x_0 \in \mathbb{R}^{m+1}_+\backslash\Sigma_{\text{int}}$ and the discussion in Section 3.2 after Theorem 3.2.0.1 implies that v is smooth in an open ball centred at x_0 and contained in $\mathbb{R}^{m+1}_+\backslash\Sigma_{\text{int}} \subset (\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$. If $x_0 \in \mathcal{O}$ then $x_0 \in \mathcal{O}\backslash\Sigma_{\text{bdry}}$ and $\Theta^{\beta}_{v}(x_0) < \varepsilon$ which, combined with the fact that $\mathcal{O}\backslash\Sigma_{\text{bdry}}$ is open in $\mathcal{O}$, implies there exists an $R > 0$ such that $R^{1-m-\beta} \int_{B^+_R(x_0)} x^{\beta}_{m+1} |\nabla v|^2 \mathrm{d}x \leq \varepsilon$, $R \leq 1$ and $\overline{\partial^0 B^+_R(x_0)} \subset \mathcal{O}\backslash\Sigma_{\text{bdry}}$. Consequently, Theorem 5.6.0.1 implies that there are $\theta, \gamma \in (0, 1)$ such that $\partial_i v \in C^{0,\gamma}(\overline{B^+_{\theta R}(x_0)}; \mathbb{R}^n)$ for $i = 1, \ldots, m$. However, we also know that $(\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$ is open in $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$. Hence there exists $\tilde{R} > 0$ such that $B^+_{\tilde{R}}(x_0) \cup \partial^0 B^+_{\tilde{R}}(x_0) \subset (\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$. Setting $r = \min\{\theta R, \tilde{R}\}$ we conclude $\partial_i v \in C^{0,\gamma}(\overline{B^+_r(x_0)}; \mathbb{R}^n)$ and $B^+_r(x_0) \cup \partial^0 B^+_r(x_0) \subset (\mathbb{R}^{m+1}_+ \cup \mathcal{O})\backslash\Sigma$ which completes the proof. □

Chapter 6

Fractional Harmonic Maps

We define and analyse a family of functionals whose critical points are Fractional Harmonic Maps. Our goal is to connect the variational problem for the functionals defined in this section to the variational problem for the energies E^β defined in Chapter 3. This will permit us to apply our regularity theory for minimisers of the energy to minimising Fractional Harmonic Maps. We assume the conditions on m, β specified in Remark 2.2.1.1 hold throughout this chapter.

Let $\mathcal{O} \subset \partial\mathbb{R}^{m+1}_+$ be open and such that a continuous linear trace operator $T : \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n) \to L^p(\mathcal{O};\mathbb{R}^n)$ exists as in the discussion in Section 2.3.2, where $p = p(\beta)$. Define

$$I^\beta(u) = \inf\{E^\beta(v) : v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;N), Tv = u\} \tag{6.0.1}$$

for $u \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;N))$. The functional I^β serves as an intrinsic energy for u in its domain, as described in Chapter 1. We note that I^0 coincides with the functional I defined by (1.0.2). Minimisers of I^β are defined as follows.

Definition 6.0.0.1. Let $u \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;N))$. We say that u minimises I^β if for every compact $K \subset \mathcal{O}$ and every $\tilde{u} \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;N))$ with $u|_{\mathcal{O}\setminus K} = \tilde{u}|_{\mathcal{O}\setminus K}$ we have $I^\beta(u) \le I^\beta(\tilde{u})$. A minimiser of I^β will be called an intrinsic minimising $\frac{1-\beta}{2}$-harmonic map. For convenience, as we consider no other kind of fractional harmonic map, we drop the prefixes intrinsic and minimising. Any $\frac{1-\beta}{2}$-harmonic map will also be broadly referred to as a fractional harmonic map.

First we show that for every $u \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;N))$, the intrinsic energy $I^\beta(u)$ is attained by some $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;N)$. To this end we make the following definition.

Definition 6.0.0.2. Let $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;N)$. We say that v is a minimal harmonic map if $E^\beta(v) \le E^\beta(w)$ for all $w \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;N)$ with $Tw = Tv$.

Lemma 6.0.0.1. *Suppose $\mathcal{O}$ is open and bounded. Let $u \in T(\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_{+}; N))$. Then there exists a minimal harmonic map v with $Tv = u$.*

Proof. We recall, see Section 2.2, that $\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_{+}; \mathbb{R}^n)$ is a Hilbert space and thus a reflexive Banach space. Furthermore, E^{β} is weakly lower semi-continuous because it is the square of a norm on a Hilbert space, and it is coercive by definition. Consider the set

$$S = \{v \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_{+}; N) : Tv = u\}$$

which is non-empty by definition. If we can show this set is sequentially weakly closed then the direct method applies and E^{β} attains a minimum in S; that is, we can find a minimal harmonic map $v \in \dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_{+}; N)$ with $Tv = u$. Hence suppose $(v_k)_{k\in\mathbb{N}}$ is a sequence in S with $v_k \rightharpoonup v$ in $\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_{+}; \mathbb{R}^n)$. We want to show $v \in S$, that is, $v(x) \in N$ for almost every $x \in \mathbb{R}^{m+1}_{+}$ and $Tv = u$.

First we show that we must have $v(x) \in N$ for almost every $x \in \mathbb{R}^{m+1}_{+}$. Note that weakly convergent sequences are bounded in $\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_{+}; \mathbb{R}^n)$. Now observe that in view of Lemmata 2.2.1.1, 2.3.5.1 and 2.3.5.2, for every bounded open $\Omega \subset \mathbb{R}^{m+1}_{+}$ we have a compact embedding $\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_{+}; \mathbb{R}^n) \hookrightarrow W^{1,2}_{\beta}(\Omega; \mathbb{R}^n) \hookrightarrow L^2_{\beta}(\Omega; \mathbb{R}^n)$. We write $\mathbb{R}^{m+1}_{+}$ as a countable union of sets Ω_n and deduce that there exist subsequences $(v_k)_{k\in\Lambda_n}$ of $(v_k)_{k\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, Λ_n is an infinite set, $\Lambda_{n+1} \subset \Lambda_n$ and $(v_k)_{k\in\Lambda_n}$ converges to v pointwise almost everywhere in Ω_n. Hence, we may choose a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ with $k_n \in \Lambda_n$ for every n. It follows that $v_{k_n} \to v$ pointwise almost everywhere in $\mathbb{R}^{m+1}_{+}$ and since each v_{k_n} satisfies $v_{k_n}(x) \in N$ for almost every $x \in \mathbb{R}^{m+1}_{+}$, we find $v(x) \in N$ for almost every $x \in \mathbb{R}^{m+1}_{+}$.

To conclude, we must show that $Tv = u$. As T is continuous, linear and its domain and codomain are Banach spaces, we have $Tv_k \rightharpoonup Tv$. However, $\{Tv_k\}_{k\in\mathbb{N}}$ is a constant sequence with $Tv_k = u$ for every k. Hence Tv_k converges strongly and thus weakly to u. The uniqueness of weak limits therefore yields $Tv = u$. It follows that S is sequentially weakly closed and the direct method applies so the lemma is proved. □

An identical calculation to the derivation of the Euler-Lagrange equations for E^{β}, see Section 3.1.1, shows that any minimal harmonic map v satisfies

$$\int_{\mathbb{R}^{m+1}_{+}} x_{m+1}^{\beta} \left(\langle \nabla v, \nabla \psi \rangle - \langle \psi, A(v)(\nabla v, \nabla v) \rangle\right) \mathrm{d}x = 0 \tag{6.0.2}$$

for every $\psi \in C^{\infty}_0(\mathbb{R}^{m+1}_{+}; \mathbb{R}^n)$. Here we may not allow ψ with non-zero boundary values in view of the boundary condition $Tv = u$. In other words, v is a weak solution of

$$\operatorname{div}(x_{m+1}^{\beta}\nabla v) + x_{m+1}^{\beta}A(v)(\nabla v, \nabla v) = 0 \quad \text{in } \mathbb{R}_{+}^{m+1}.$$

In order to analyse variations of I^{β} we must consider the left hand side of (6.0.2) for ψ with non-zero boundary values, in which case this integral is no longer necessarily zero for a given minimal harmonic map v. We observe the following.

Lemma 6.0.0.2. *Let $u \in T(\dot{W}_{\beta}^{1,2}(\mathbb{R}_{+}^{m+1}; N))$, $v \in \dot{W}_{\beta}^{1,2}(\mathbb{R}_{+}^{m+1}; N)$ be a minimial harmonic map with $Tv = u$ and $\psi \in \mathcal{D}_{+}(\mathbb{R}_{+}^{m+1}; \mathbb{R}^{n})$ with $\psi(x', 0) = \phi(x')$ for $\phi \in C_0^{\infty}(\mathcal{O}; \mathbb{R}^{n})$. Then the integral*

$$\int_{\mathbb{R}_{+}^{m+1}} x_{m+1}^{\beta} \left(\langle \nabla v, \nabla \psi \rangle - \langle \psi, A(v)(\nabla v, \nabla v) \rangle \right) \mathrm{d}x \tag{6.0.3}$$

only depends on ϕ.

In order to prove this lemma, we will consider the integral (6.0.3) for $\psi \in \mathcal{D}_{+}(\mathbb{R}_{+}^{m+1}; \mathbb{R}^{n})$ with $\psi(x', 0) = \phi(x') = 0$. In particular, we will show that any such ψ may be approximated by a sequence of $C_0^{\infty}(\mathbb{R}_{+}^{m+1}; \mathbb{R}^{n})$ functions in a suitable sense. To this end, recall the cutoff function $\chi \in C_0^{\infty}(\mathbb{R}; [0, 1])$ defined by (3.3.1) in Section 3.3.1:

$$\chi(s) = \begin{cases} 0 & s \in (-\infty, \frac{1}{2}) \\ 0 \le \chi(s) \le 1 & s \in [\frac{1}{2}, 1) \\ 1 & s \in [1, \infty). \end{cases}$$

For $\delta > 0$ define $\chi_{\delta}(x_{m+1}) = \chi(\frac{x_{m+1}}{\delta})$. The pointwise limit as $\delta \searrow 0$ is

$$\chi(s) = \begin{cases} 0 & x_{m+1} \in (-\infty, 0) \\ 1 & x_{m+1} \in [0, \infty) \end{cases}$$

and $(\chi_{\delta})'(x_{m+1}) = \frac{1}{\delta}\chi'(\frac{x_{m+1}}{\delta}) = 0$ outside the interval $[\frac{\delta}{2}, \delta]$.

Lemma 6.0.0.3. *Let $\psi \in \mathcal{D}_{+}(\mathbb{R}_{+}^{m+1}; \mathbb{R}^{n})$ with $\psi(x', 0) = 0$. Then the sequence $(\psi_k)_{k \in \mathbb{N}}$ defined by $\psi_k = \chi_{\frac{1}{k}}\psi \in C_0^{\infty}(\mathbb{R}_{+}^{m+1}; \mathbb{R}^{n})$ satisfies $\psi_k \to \psi$ in $\dot{W}_{\beta}^{1,2}(\mathbb{R}_{+}^{m+1}; \mathbb{R}^{n})$ and uniformly in $\mathbb{R}_{+}^{m+1}$.*

Proof. Consider the sequence $(\psi_k)_{k \in \mathbb{N}} = (\chi_{\frac{1}{k}}\psi)_{k \in \mathbb{N}}$. By definition we have $\psi_k \in$

$C_0^\infty(\mathbb{R}_+^{m+1};\mathbb{R}^n)$ for every k. First we show $\psi_k \to \psi$ in $\dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1};\mathbb{R}^n)$. We have

$$\int_{\mathbb{R}_+^{m+1}} x_{m+1}^\beta |\nabla(\chi_{\frac{1}{k}}\psi) - \nabla\psi|^2 \mathrm{d}x \le 2\int_{\mathbb{R}_+^{m+1}} x_{m+1}^\beta (1-\chi_{\frac{1}{k}})^2 |\nabla\psi|^2 \mathrm{d}x + 2\int_{\mathbb{R}_+^{m+1}} x_{m+1}^\beta (\chi'_{\frac{1}{k}})^2 |\psi|^2 \mathrm{d}x. \tag{6.0.4}$$

We consider the terms on the right hand side of (6.0.4). Observe that $(1-\chi_{\frac{1}{k}})^2 \to 0$ pointwise and $x_{m+1}^\beta(1-\chi_{\frac{1}{k}})^2|\nabla\psi|^2 \le x_{m+1}^\beta|\nabla\psi|^2 \in L^1(\mathbb{R}_+^{m+1};\mathbb{R}^n)$. Hence, Lebesgue's Dominated Convergence Theorem implies that

$$\int_{\mathbb{R}_+^{m+1}} x_{m+1}^\beta (1-\chi_{\frac{1}{k}})^2 |\nabla\psi|^2 \mathrm{d}x \to 0 \text{ as } k \to \infty. \tag{6.0.5}$$

Now we consider the remaining term in (6.0.4). We integrate with respect to x_{m+1}, using the fact that $\psi(x',0)=0$ for every $x' \in \mathbb{R}^m$, to see that

$$\int_{\mathbb{R}_+^{m+1}} x_{m+1}^\beta (\chi'_{\frac{1}{k}})^2 |\psi|^2 \mathrm{d}x \le \int_{\mathbb{R}^m}\int_0^{\frac{1}{k}} x_{m+1}^\beta (\chi'_{\frac{1}{k}})^2 \left|\int_0^{x_{m+1}} \frac{\partial\psi}{\partial x_{m+1}}\mathrm{d}s\right|^2 \mathrm{d}x_{m+1}\mathrm{d}x'. \tag{6.0.6}$$

Using Hölder's inequality followed by Fubini's theorem we find

$$\begin{aligned}
&\int_{\mathbb{R}^m}\int_0^{\frac{1}{k}} x_{m+1}^\beta (\chi'_{\frac{1}{k}})^2 \left|\int_0^{x_{m+1}} \frac{\partial\psi}{\partial x_{m+1}}\mathrm{d}s\right|^2 \mathrm{d}x_{m+1}\mathrm{d}x' \\
&= \int_{\mathbb{R}^m}\int_0^{\frac{1}{k}} x_{m+1}^\beta (\chi'_{\frac{1}{k}})^2 \left|\int_0^{x_{m+1}} s^{-\frac{\beta}{2}} s^{\frac{\beta}{2}} \frac{\partial\psi}{\partial x_{m+1}}\mathrm{d}s\right|^2 \mathrm{d}x_{m+1}\mathrm{d}x' \\
&\le C\int_{\mathbb{R}^m}\int_0^{\frac{1}{k}} x_{m+1} (\chi'_{\frac{1}{k}})^2 \int_0^{x_{m+1}} s^\beta \left|\frac{\partial\psi}{\partial x_{m+1}}\right|^2 \mathrm{d}s\mathrm{d}x_{m+1}\mathrm{d}x' \\
&\le C\int_{\mathbb{R}^m}\int_0^{\frac{1}{k}} |\chi'_{\frac{1}{k}}|\mathrm{d}x_{m+1} \int_0^{\frac{1}{k}} s^\beta \left|\frac{\partial\psi}{\partial x_{m+1}}\right|^2 \mathrm{d}s\mathrm{d}x' \\
&\le C\int_{\mathbb{R}^m}\int_0^{\frac{1}{k}} s^\beta \left|\frac{\partial\psi}{\partial x_{m+1}}\right|^2 \mathrm{d}s\mathrm{d}x'.
\end{aligned} \tag{6.0.7}$$

Since $s^\beta\left|\frac{\partial\psi}{\partial x_{m+1}}\right|^2 \mathbb{1}_{\mathbb{R}^m\times[0,\frac{1}{k}]} \le s^\beta\left|\frac{\partial\psi}{\partial x_{m+1}}\right|^2 \in L^1(\mathbb{R}_+^{m+1};\mathbb{R}^n)$ we may use Lebesgue's Dominated Convergence Theorem to see that

$$\int_{\mathbb{R}^m}\int_0^{\frac{1}{k}} s^\beta \left|\frac{\partial\psi}{\partial x_{m+1}}\right|^2 \mathrm{d}s\mathrm{d}x' \to 0 \text{ as } k \to \infty. \tag{6.0.8}$$

Hence, combining (6.0.4), (6.0.5), (6.0.6), (6.0.7) and (6.0.8) we see that

$$\int_{\mathbb{R}^{m+1}_+} x_{m+1}^{\beta} |\nabla(\chi_{\frac{1}{k}}\psi) - \nabla\psi|^2 \mathrm{d}x \to 0 \text{ as } k \to \infty$$

as required.

Now we will show that $\psi_k \to \psi$ uniformly on $\mathbb{R}^{m+1}_+$. To see this, observe that $\psi_k = \psi$ on $\mathbb{R}^m \times [\frac{1}{k}, \infty)$ and hence

$$\sup_{x \in \mathbb{R}^{m+1}_+} |\psi_k(x) - \psi(x)| = \sup_{x \in \mathbb{R}^m \times [0, \frac{1}{k}]} |\psi_k(x) - \psi(x)| \le \sup_{x \in \mathbb{R}^m \times [0, \frac{1}{k}]} |\psi(x)|. \quad (6.0.9)$$

Since $\psi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ it follows that $\nabla\psi$ is bounded in $\mathbb{R}^{m+1}_+$. We combine this fact with the assumption $\psi(x', 0) = 0$ for every $x' \in \mathbb{R}^m$ to see that

$$|\psi(x)| \le \int_0^{x_{m+1}} \left| \frac{\partial \psi}{\partial x_{m+1}} \right| \mathrm{d}s \le \int_0^{\frac{1}{k}} \left| \frac{\partial \psi}{\partial x_{m+1}} \right| \mathrm{d}s \le \frac{1}{k} ||\nabla\psi||_{L^\infty(\mathbb{R}^{m+1}_+; \mathbb{R}^{(m+1)n})} \quad (6.0.10)$$

for every $x \in \mathbb{R}^m \times [0, \frac{1}{k}]$. Together (6.0.9) and (6.0.10) imply

$$\sup_{x \in \mathbb{R}^{m+1}_+} |\psi_k(x) - \psi(x)| \to 0 \text{ as } k \to \infty$$

which concludes the proof. □

Proof of Lemma 6.0.0.2. Let v be minimal harmonic map with $Tv = u$. Then for every $\psi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ we calculate

$$\begin{aligned} &\left| \int_{\mathbb{R}^{m+1}_+} x_{m+1}^{\beta} \left(\langle \nabla v, \nabla \psi \rangle - \langle \psi, A(v)(\nabla v, \nabla v) \rangle \right) \mathrm{d}x \right| \\ &\le ||v||_{\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; \mathbb{R}^n)} ||\psi||_{\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; \mathbb{R}^n)} + C||\psi||_{L^\infty(\mathbb{R}^{m+1}_+; \mathbb{R}^n)} ||v||^2_{\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_+; \mathbb{R}^n)} \end{aligned} \quad (6.0.11)$$

for a constant C depending on N. Now let $\phi \in C_0^\infty(\mathcal{O}; \mathbb{R}^n)$ and suppose $\tilde{\psi}, \hat{\psi} \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ satisfy $\tilde{\psi}(x', 0) = \hat{\psi}(x', 0) = \phi(x')$ for every $x' \in \mathbb{R}^n$. Define $\psi = \tilde{\psi} - \hat{\psi}$. Then $\psi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ and $\psi(x', 0) = 0$ for every $x' \in \mathbb{R}^m$. Let $(\psi_k)_{k \in \mathbb{N}}$ be the approximating sequence given by Lemma 6.0.0.3 with $\psi_k \in C_0^\infty(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ for every k. Then

$$\int_{\mathbb{R}^{m+1}_+} x_{m+1}^{\beta} \left(\langle \nabla v, \nabla \psi_k \rangle - \langle \psi_k, A(v)(\nabla v, \nabla v) \rangle \right) \mathrm{d}x = 0$$

for every k by (6.0.2). Furthermore, since $\psi_k \to \psi$ both in $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ and uniformly on $\mathbb{R}^{m+1}_+$, using (6.0.11) we see that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} \left(\langle \nabla v, \nabla(\psi_k - \psi) \rangle - \langle (\psi_k - \psi), A(v)(\nabla v, \nabla v) \rangle \right) \mathrm{d}x \right| \\
&\leq ||v||_{\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)} ||\psi_k - \psi||_{\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)} + C||\psi_k - \psi||_{L^\infty(\mathbb{R}^{m+1}_+;\mathbb{R}^n)} ||v||^2_{\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;\mathbb{R}^n)} \\
&\to 0 \text{ as } k \to \infty. \qquad (6.0.12)
\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} \left(\langle \nabla v, \nabla \psi \rangle - \langle \psi, A(v)(\nabla v, \nabla v) \rangle \right) \mathrm{d}x = 0$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} \left(\langle \nabla v, \nabla \psi_1 \rangle - \langle \psi_1, A(v)(\nabla v, \nabla v) \rangle \right) \mathrm{d}x \\
&= \int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} \left(\langle \nabla v, \nabla \psi_2 \rangle - \langle \psi_2, A(v)(\nabla v, \nabla v) \rangle \right) \mathrm{d}x.
\end{aligned}$$

It follows that the integral (6.0.3) only depends on ϕ as required. □

A formal calculation suggests that if $\psi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+;\mathbb{R}^n)$ with $\psi(x',0) = \phi(x')$ for $\phi \in C^\infty_0(\mathcal{O};\mathbb{R}^n)$, then for any minimal harmonic map v we have

$$\begin{aligned}
&\int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} \left(\langle \psi, A(v)(\nabla v, \nabla v) \rangle - \langle \nabla v, \nabla \psi \rangle \right) \mathrm{d}x \\
&= \int_{\mathcal{O}} \left\langle \lim_{x_{m+1} \to 0^+} x^\beta_{m+1} \frac{\partial v}{\partial x_{m+1}}(x', x_{m+1}), \phi(x') \right\rangle \mathrm{d}x'.
\end{aligned}$$

This will only be the case if v has sufficient regularity in $\mathbb{R}^{m+1}_+ \cup \mathcal{O}$. However, as a consequence of Lemma 6.0.0.2 we see that the integral (6.0.3) defines a distribution on $\mathcal{O}$. We write

$$\partial^\beta_{m+1} v(\phi) = \int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} \left(\langle \psi, A(v)(\nabla v, \nabla v) \rangle - \langle \nabla v, \nabla \psi \rangle \right) \mathrm{d}x \qquad (6.0.13)$$

for every $\phi \in C^\infty_0(\mathcal{O};\mathbb{R}^n)$. This observation allows us, analogously to [34] Proposition 1.1, to identify a superdifferential for I^β. Recall that π_N denotes the nearest point projection onto N.

Proposition 6.0.0.1. *Let $u \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;N))$ and $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+;N)$ be a* *minimal harmonic map with $Tv = u$. Then for $\phi \in C_0^\infty(\mathcal{O};\mathbb{R}^n)$,*

$$I^\beta(\pi_N(u+t\phi)) \leq I^\beta(u) - t\partial_{m+1}^\beta v(\phi) + o(|t|) \tag{6.0.14}$$

for $t \to 0$.

Proof. Let $u_t = \pi_N(u + t\phi)$ and $v_t = \pi_N(v + t\psi)$ for sufficiently small t and some $\psi \in \mathcal{D}_+(\mathbb{R}_+^{m+1};\mathbb{R}^n)$ with $\psi(x',0) = \phi(x')$. We have $Tv_t = u_t$. Thus $I^\beta(u_t) \leq E^\beta(v_t)$ and, by assumption, $I^\beta(u) = E^\beta(v)$. Our calculation of the Euler-Lagrange equations for E^β in Section 3.1.1 shows that

$$\left.\frac{\partial}{\partial t}\right|_{t=0} E^\beta(v_t) = -\partial_{m+1}^\beta v(\phi).$$

We combine this fact with the definition of the derivative to see that

$$\begin{aligned} I^\beta(\pi_N(u+t\phi)) &\leq E^\beta(v_t) \\ &= E^\beta(v) - t\partial_{m+1}^\beta v(\phi) + o(|t|) \\ &= I^\beta(u) - t\partial_{m+1}^\beta v(\phi) + o(|t|) \end{aligned}$$

as required. □

Remark 6.0.0.1. Suppose the derivative $\left.\frac{\partial}{\partial t}\right|_{t=0} I^\beta(\pi_N(u+t\phi))$ exists. When $t > 0$ the preceding proposition implies that

$$\lim_{t\to 0^+} \frac{I^\beta(\pi_N(u+t\phi)) - I^\beta(u)}{t} \leq -\partial_{m+1}^\beta v(\phi).$$

If $t < 0$ then

$$\lim_{t\to 0^-} \frac{I^\beta(\pi_N(u+t\phi)) - I^\beta(u)}{t} \geq -\partial_{m+1}^\beta v(\phi).$$

Hence, whenever $\left.\frac{\partial}{\partial t}\right|_{t=0} I^\beta(\pi_N(u+t\phi))$ exists we have

$$\left.\frac{\partial}{\partial t}\right|_{t=0} I^\beta(\pi_N(u+t\phi)) = -\partial_{m+1}^\beta v(\phi)$$

for any minimal harmonic map v with $Tv = u$.

The above considerations suggest that we may now be able to calculate the Euler-Lagrange equation for I^β. To this end we make the following definition.

Definition 6.0.0.3. Let $\beta \in (-1,1)$. Define $\mathcal{D}_\beta$ as the collection of all $u \in T(\dot{W}_\beta^{1,2}(\mathbb{R}_+^{m+1};N))$ such that there exists a distribution $\lambda_\beta \in (C_0^\infty(\mathcal{O};\mathbb{R}^n))^*$ with

$\lambda_\beta = -\partial^\beta_{m+1} v$ for every minimal harmonic map $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ with $Tv = u$. Then we may define a map $\Lambda_\beta : \mathcal{D}_\beta \to (C_0^\infty(\mathcal{O}; \mathbb{R}^n))^* : u \mapsto \lambda_\beta = \Lambda_\beta u$.

The map Λ_β is a Dirichlet to Neumann map for the harmonic map problem: fix $u \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N))$ and minimise E^β in $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ among all maps with trace u. In other words, it sends the Dirichlet data u to the Neumann data $-\partial^\beta_{m+1} v$. In [34] Theorem 1.1 Moser showed that for $\beta = 0$, this map is actually the first variation of I_0. In an almost identical way, we can prove the following.

Theorem 6.0.0.1. *Let $u \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N))$. If $u \in \mathcal{D}_\beta$, then*

$$\frac{\partial}{\partial t}\bigg|_{t=0} I^\beta(\pi_N(u + t\phi)) = \Lambda_\beta u(\phi)$$

for all $\phi \in C_0^\infty(\mathcal{O}; \mathbb{R}^n)$. If $u \notin \mathcal{D}_\beta$, then there exists $\phi \in C_0^\infty(\mathcal{O}; \mathbb{R}^n)$ such that the function $t \mapsto I^\beta(\pi_N(u + t\phi))$ is not differentiable at 0.

We give the proof for completeness, as an explanation of ideas involved and verification that the method of proof in [34] is valid in the context considered here.

Proof. The fact that $t \mapsto I^\beta(\pi_N(u+t\phi))$ is not differentiable at 0 when $u \notin \mathcal{D}_\beta$ is a consequence of Proposition 6.0.0.1, in particular it follows from remark 6.0.0.1. This proposition also shows that

$$I^\beta(u_t) \leq I^\beta(u) + t\Lambda_\beta u + o(|t|) \tag{6.0.15}$$

where $u_t = \pi_N(u + t\phi)$ and t is small. To complete the proof of the theorem we will construct a minimal harmonic map $w \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ which we will use to show the reverse inequality to (6.0.14) in Proposition 6.0.0.1 is satisfied; that is

$$I^\beta(u_t) \geq I^\beta(u) + t\Lambda_\beta u + o(|t|). \tag{6.0.16}$$

It then follows from (6.0.15) that (6.0.16)

$$t\Lambda_\beta u + o(|t|) \leq I^\beta(u_t) - I^\beta(u) \leq t\Lambda_\beta u + o(|t|)$$

which shows that $\frac{\partial}{\partial t}\big|_0 I^\beta(u_t) = \Lambda^\beta u$.

Let $u \in \mathcal{D}_\beta$, then for every minimal harmonic map $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ with $Tv = u$ we have $\Lambda_\beta u = -\partial^\beta_{m+1} v$. Fix such a v and consider the variations of u and v, respectively given by $u_t = \pi_N(u + t\phi)$ and $v_t = \pi_N(v + t\psi)$ for some

$\psi \in \mathcal{D}_+(\mathbb{R}^{m+1}_+; \mathbb{R}^n)$ and $\phi \in C_0^\infty(\mathcal{O}; \mathbb{R}^n)$ with $\psi(x', 0) = \phi(x')$. We may assume $t \in [-1, 1]$ since we are only interested in the variations v_t and u_t for t near 0. Let w_t be a minimal harmonic map with $Tw_t = Tv_t = u_t$, which exists in view of Lemma 6.0.0.1. We have $Tw_t = u_t$, $I^\beta(u_t) = E^\beta(w_t) \leq E^\beta(v_t)$ and $I^\beta(u) = E^\beta(v)$. Bearing these facts in mind, our goal is now to show that as $t \to 0$, w_t converges (in an appropriate sense) to a minimal harmonic map w with $Tw = u$ and which we can use to show (6.0.16) holds.

To further motivate this aim and outline the strategy for the proof, we observe the following. For small t, the variations $u_t = \pi_N(u + t\phi)$ and $u_{-t} = \pi_N(u - t\phi)$ are approximately inverse to each other; that is, $\pi_N(u_t - t\phi)$ and $\pi_N(u_{-t} + t\phi)$ are approximately equal to u (they are equal to u when $t = 0$). We also have $Tw_t = u_t$ and $T\pi_N(w_t - t\psi) = T\pi_N(u_t - t\psi)$. If w_t was both minimal harmonic and of the form $w_t = \pi_N(w + t\psi)$ for a minimal harmonic map w with $Tw = u$, then by the definition of the derivative, using the same method as in the proof of Proposition 6.0.0.1, we would have

$$I^\beta(u_t) = E^\beta(w_t) = E^\beta(w) - t\partial^\beta_{m+1} w + o(|t|) = I^\beta(u) + t\Lambda_\beta u + o(|t|)$$

as required. Unfortunately, w_t need not have this form. However, as we intend to show that w_t converges to a minimal harmonic map w with $Tw = u$, we still expect $\pi_N(w_t - t\phi)$ to approximate such a w. By considering the error in the aforementioned approximation, we will show that (6.0.16) holds.

We examine the map $y \mapsto \pi_N(y + \eta)$, where $y \in N$ and $\eta \in \mathbb{R}^n$ is fixed with $|\eta|$ sufficiently small (depending on N) so that the map is well defined, in more detail. As suggested above, for such η, the inverse in the y variable is approximately $y \mapsto \pi_N(y - \eta)$; the following facts about this approximation have been proven already in [34] section 2. Define the map

$$\Xi_\eta : N \to N : y \to \pi_N(\pi_N(y + \eta) - \eta).$$

For sufficiently small η the inverse, with respect to y, exists and we write

$$\Theta_\eta = \Xi_\eta^{-1}.$$

Furthermore, as stated in [34] section 2, there exists a constant $C = C(N)$ such that

$$|\Xi_\eta(y) - y| \leq C|\eta|^2 \quad \text{and} \quad |D_\eta \Xi_\eta(y)| \leq C|\eta| \tag{6.0.17}$$

and

$$|\Theta_\eta(y) - y| \leq C|\eta|^2 \quad \text{and} \quad |D_\eta \Theta_\eta(y)| \leq C|\eta|. \tag{6.0.18}$$

and, for a tangent vector $Y \in T_y N$,

$$|D_y \Xi_\eta(y) Y - Y| \leq C|\eta|^2 |Y| \quad \text{and} \quad |D_y \Theta_\eta(y) Y - Y| \leq C|\eta|^2 |Y|. \tag{6.0.19}$$

Now we proceed with our calculations. First we show that the family of maps $\{w_t\}_{t\in[-1,1]}$ is bounded, independently of t, in $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$. As a consequence of the Mean Value Theorem, we have

$$E^\beta(v_t) \leq E^\beta(v) + |t| \left| \frac{\partial}{\partial t}\bigg|_{t=\tau} E^\beta(v_t) \right| \tag{6.0.20}$$

for some $\tau \in (-t, t)$. Furthermore, it follows from (3.1.2) in Section 3.1.1 that

$$\begin{aligned}
&\left| \frac{\partial}{\partial t}(x^\beta_{m+1} |\nabla v_t|^2) \right| \\
&\leq 2x^\beta_{m+1} \left| \sum_{i=1}^{m+1} \left\langle d\pi_N(v + t\psi) \left(\frac{\partial v}{\partial x_i} + t\frac{\partial \psi}{\partial x_i} \right), \mathrm{Hess}\pi_N(v + t\psi) \left(\frac{\partial v}{\partial x_i} + t\frac{\partial \psi}{\partial x_i}, \psi \right) \right\rangle \right| \\
&\quad + 2x^\beta_{m+1} \left| \sum_{i=1}^{m+1} \left\langle d\pi_N(v + t\psi) \left(\frac{\partial v}{\partial x_i} + t\frac{\partial \psi}{\partial x_i} \right), d\pi_N(v + t\psi) \left(\frac{\partial \psi}{\partial x_i} \right) \right\rangle \right| \\
&\leq C x^\beta_{m+1} |\nabla v + t\nabla \psi|^2 \, |\psi|_{L^\infty(\mathbb{R}^{m+1}_+;\mathbb{R}^n)} + C x^\beta_{m+1} |\nabla v + t\nabla\psi| \, |\nabla \psi|.
\end{aligned} \tag{6.0.21}$$

Henceforth we allow the constants in subsequent inequalities to depend on positive powers of $|\psi|_{L^\infty(\mathbb{R}^{m+1}_+;\mathbb{R}^n)}$. We combine (6.0.21) with an application of Young's inequality to see that

$$\left| \frac{\partial}{\partial t}(x^\beta_{m+1} |\nabla v_t|^2) \right| \leq C x^\beta_{m+1} \left(|\nabla v + t\nabla\psi|^2 + |\nabla\psi|^2 \right) \leq C x^\beta_{m+1} (|\nabla v|^2 + |\nabla \psi|^2). \tag{6.0.22}$$

We now allow C to depend on $E^\beta(\psi)$ as well. Then together, (6.0.20) and (6.0.22) imply

$$E^\beta(w_t) \leq E^\beta(v_t) \leq E^\beta(v) + |t| C(E^\beta(v) + 1) = I^\beta(u) + |t| C(I^\beta(u) + 1), \tag{6.0.23}$$

which shows that $\{w_t\}_{t\in[-1,1]}$ is bounded, independently of t, in $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$.

Now we consider the energy of $\tilde{w}_t = \pi_N(w_t - t\psi)$. We will bound $|\nabla \tilde{w}_t|^2$ above in terms of the gradient of w_t and terms which, upon integration, constitute $\partial^\beta_{m+1} w_t$. Furthermore, we will use the maps Ξ_η and Θ_η and the corresponding estimates above to construct a family of maps $\{\hat{w}_t\}_{t\in[-1,1]}$, all with trace equal to u, such that the difference between the energy of $\hat{w}_t$ and $\tilde{w}_t$ is $o(|t|)$. Define

$$\tilde{u}_t = \pi_N(u_t - t\phi) = \Xi_{t\phi}(u)$$

and notice that $T\tilde{w}_t = \tilde{u}_t$. Then $\Theta_{t\phi}(\tilde{u}_t) = u$ and the maps

$$\hat{w}_t = \Theta_{t\psi}(\tilde{w}_t)$$

all satisfy

$$T\hat{w}_t = \Theta_{T(t\psi)}(T\tilde{w}_t) = \Theta_{t\phi}(\tilde{u}_t) = \Theta_{t\phi} \circ \Xi_{t\phi}(u) = u.$$

We proceed to calculate the gradients and bound the energies of $\hat{w}_t$ and $\tilde{w}_t$. In what follows we use D to denote differentiation in $\mathbb{R}^n$. Lemma 3.1 of [33] states that $D\pi_N(w_t)$ is the orthogonal projection onto $T_{w_t}N$ since $w_t \in N$. Hence we calculate

$$\begin{aligned}
\nabla \tilde{w}_t &= D\pi_N(w_t - t\psi)(\nabla w_t - t\nabla\psi) \\
&= \nabla w_t + (D\pi_N(w_t - t\psi) - D\pi_N(w_t))\nabla w_t - tD\pi_N(w_t - t\psi)\nabla\psi. \quad (6.0.24)
\end{aligned}$$

We add and subtract $tD^2\pi_N(w_t)(\nabla w_t, \psi)$ and $tD\pi_N(w_t)\nabla\psi$ from (6.0.24) and observe that

$$\begin{aligned}
|\nabla \tilde{w}_t|^2 = |\nabla w_t|^2 &- 2t\langle \nabla w_t, D^2\pi_N(w_t)(\nabla w_t, \psi)\rangle - 2t\langle \nabla w_t, D\pi_N(w_t)\nabla\psi\rangle \\
&- 2t\langle \nabla w_t, (D\pi_N(w_t - t\psi) - D\pi_N(w_t))\nabla\psi\rangle \\
&+ 2\langle \nabla w_t, (D\pi_N(w_t - t\psi) - D\pi_N(w_t))\nabla w_t + tD^2\pi_N(w_t)(\nabla w_t, \psi)\rangle \\
&+ |(D\pi_N(w_t - t\psi) - D\pi_N(w_t))\nabla w_t - tD\pi_N(w_t - t\phi)\nabla\psi|^2. \quad (6.0.25)
\end{aligned}$$

We now expand this expression into $|\nabla w_t|^2$ plus terms constituting the integrand of $\partial^\beta_{m+1} w_t$ and a remainder of $o(|t|)$. For almost every $x \in \mathbb{R}^{m+1}_+$, the second fundamental form $A(w_t) \in (T_{w_t}N)^\perp$ and $\frac{\partial w_t}{\partial x_i} \in T_{w_t}N$, where $i = 1, \ldots, m+1$. Hence we deduce from lemma 3.2 in [33] that

$$\langle \nabla w_t, D^2\pi_N(w_t)(\nabla w_t, \psi)\rangle = -\langle A(w_t)(\nabla w_t, \nabla w_t), \psi\rangle. \quad (6.0.26)$$

Similarly, as $\frac{\partial w_t}{\partial x_i} \in T_{w_t}N$, where $i = 1, \ldots, m+1$, and $D\pi_N(w_t)$ is the projection onto $T_{w_t}N$, we have

$$\langle \nabla w_t, D\pi_N(w_t)\nabla\psi\rangle = \langle \nabla w_t, \nabla\psi\rangle. \tag{6.0.27}$$

Added together, (6.0.26) and (6.0.27) comprise the integrand of $\partial^\beta_{m+1} w_t$.

The Mean Value Theorem guarantees that

$$|(D\pi_N(w_t - t\psi) - D\pi_N(w_t))\nabla\psi| \leq C|t||\nabla\psi| \tag{6.0.28}$$

and

$$|(D\pi_N(w_t - t\psi) - D\pi_N(w_t))\nabla w_t| \leq C|t||\nabla w_t|. \tag{6.0.29}$$

It also follows from two applications of the Mean Value Theorem and the boundedness of the derivatives of π_N, that there is a constant C such that

$$|(D\pi_N(w_t - t\psi) - D\pi_N(w_t))\nabla w_t + tD^2\pi_N(w_t)(\nabla w_t, \psi)| \leq Ct^2|\nabla w_t|. \tag{6.0.30}$$

We combine (6.0.26), (6.0.27), (6.0.28), (6.0.29) and (6.0.30) with (6.0.25) to see that

$$\begin{aligned}|\nabla\tilde{w}_t|^2 \leq{}& |\nabla w_t|^2 + 2t\langle A(w_t)(\nabla w_t, \nabla w_t), \psi\rangle - 2t\langle \nabla w_t, \nabla\psi\rangle \\ &+ Ct^2(|\nabla w_t|^2 + |\nabla\psi|^2).\end{aligned}$$

Multiplying by x^β_{m+1} and integrating over $\mathbb{R}^{m+1}_+$ gives

$$\begin{aligned}E^\beta(\tilde{w}_t) &\leq E^\beta(w_t) + t\partial^\beta_{m+1}w_t(\phi) + Ct^2(E^\beta(w_t) + 1) \\ &= I^\beta(u_t) + t\partial^\beta_{m+1}w_t(\phi) + Ct^2(I^\beta(u_t) + 1).\end{aligned} \tag{6.0.31}$$

To permit the comparison of the energy of $\hat{w}_t$ and $\tilde{w}_t$ we calculate

$$\nabla\hat{w}_t = D_y\Theta_{t\psi}(\tilde{w}_t)\nabla\tilde{w}_t + tD_\eta\Theta_{t\psi}(\tilde{w}_t)\nabla\psi.$$

Applying (6.0.18) and (6.0.19) we see that

$$\begin{aligned}|\nabla\hat{w}_t - \nabla\tilde{w}_t| &= |D_y\Theta_{t\psi}(\tilde{w}_t)\nabla\tilde{w}_t - \nabla\tilde{w}_t + tD_\eta\Theta_{t\psi}(\tilde{w}_t(x))\nabla\psi| \\ &\leq |D_y\Theta_{t\psi}(\tilde{w}_t)\nabla\tilde{w}_t - \nabla\tilde{w}_t| + |t||D_\eta\Theta_{t\psi}(\tilde{w}_t)\nabla\psi| \\ &= Ct^2(|\nabla\tilde{w}_t| + |\nabla\psi|).\end{aligned} \tag{6.0.32}$$

Observe that

$$E^\beta(\hat{w}_t) = E^\beta(\tilde{w}_t) + 2\int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} \langle \nabla \hat{w}_t - \nabla \tilde{w}_t, \nabla \tilde{w}_t \rangle \,\mathrm{d}x + E^\beta(\hat{w}_t - \tilde{w}_t).$$

An application of Young's inequality combined with (6.0.32) gives

$$\begin{aligned} \int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} \langle \nabla \hat{w}_t - \nabla \tilde{w}_t, \nabla \tilde{w}_t \rangle \,\mathrm{d}x &\leq \int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} |\nabla \hat{w}_t - \nabla \tilde{w}_t| \, |\nabla \tilde{w}_t| \,\mathrm{d}x \\ &\leq Ct^2 \int_{\mathbb{R}^{m+1}_+} x^\beta_{m+1} |\nabla \tilde{w}_t|^2 + |\nabla \tilde{w}_t| \, |\nabla \psi| \,\mathrm{d}x \\ &\leq Ct^2 (E^\beta(\tilde{w}_t) + 1). \end{aligned} \tag{6.0.33}$$

In a similar way we deduce that

$$E^\beta(\hat{w}_t - \tilde{w}_t) \leq Ct^2(E^\beta(\tilde{w}_t) + 1). \tag{6.0.34}$$

Hence, in view of (6.0.33) and (6.0.34), we have

$$E^\beta(\hat{w}_t) \leq E^\beta(\tilde{w}_t) + Ct^2(E^\beta(\tilde{w}_t) + 1). \tag{6.0.35}$$

Combining (6.0.31) and (6.0.35) we see that

$$I^\beta(u) \leq E^\beta(\hat{w}_t) \leq I^\beta(u_t) + t\partial^\beta_{m+1} w_t(\phi) + t^2 C(I^\beta(u_t) + 1).$$

We combine this fact with the observation $E^\beta(w_t) = I^\beta(u_t)$ and (6.0.23) to see that

$$\begin{aligned} I^\beta(u) &\leq I^\beta(u_t) + t\partial^\beta_{m+1} w_t(\phi) + Ct^2(I^\beta(u) + 1) \\ &\leq I^\beta(u_t) + t\partial^\beta_{m+1} w_t(\phi) + o(|t|). \end{aligned} \tag{6.0.36}$$

If we can replace $t\partial^\beta_{m+1} w_t(\phi)$ with $-\Lambda_\beta u(\phi)$ in (6.0.36), whilst only changing the inequality by terms of the form $o(|t|)$, then we have proved (6.0.16). Choose a sequence $(t_k)_{k\in\mathbb{N}}$ with $t_k \to 0$ as $k \to \infty$. Then $(w_{t_k})_{k\in\mathbb{N}}$ is a bounded sequence in $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ in view of (6.0.23) and so we may extract a subsequence $(t_{k_l})_{l\in\mathbb{N}}$ such that $w_{t_{k_l}} \rightharpoonup w \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ weakly. The continuity of the trace operator implies $Tw = u$. However, v is a minimal harmonic map and so in view of (6.0.23),

$$\limsup_{l\to\infty} E^\beta(w_{t_{k_l}}) \leq E^\beta(v) \leq E^\beta(w).$$

Thus $w_{t_{k_l}} \to w$ stongly in $\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ and so w is also a minimal harmonic map with $Tw = u$. Moreover, $\partial^\beta_{m+1} w$ is well defined and

$$\partial^\beta_{m+1} w(\phi) = \lim_{l\to\infty} \partial^\beta_{m+1} w_{t_{k_l}}(\phi).$$

Since $u \in \mathcal{D}_\beta$ we have $\partial^\beta_{m+1} w(\phi) = -\Lambda_\beta u(\phi)$ which means that the preceding limit does not depend on the choice of subsequence $(t_{k_l})_{l\in\mathbb{N}}$. Hence

$$\Lambda_\beta u(\phi) = -\lim_{t\to 0} \partial^\beta_{m+1} w_t(\phi).$$

It follows from (6.0.36) that as $t \to 0$ we have

$$I^\beta(u) + t\Lambda_\beta u + o(|t|) \le I^\beta(u_t)$$

which is (6.0.16). This concludes the proof. $\square$

Corollary 6.0.0.1. *Suppose $u \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N))$ is a critical point of I^β in the sense that $\frac{\partial}{\partial t}\big|_0 I^\beta(u_t) = 0$ where $u_t = \pi_N(u + t\phi)$ for $\phi \in C^\infty_0(\mathcal{O}; \mathbb{R}^n)$. Then $u \in \mathcal{D}_\beta$ and $\Lambda_\beta u = 0$.*

Now we consider the regularity of $\frac{1-\beta}{2}$-harmonic maps. To this end, we establish a link between minimisers of I^β and minimisers of E^β relative to $\mathcal{O}$.

Lemma 6.0.0.4. *Suppose $u \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N))$ minimises I^β and fix a minimal harmonic map $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ with $Tv = u$. Then v is a minimiser of E^β relative to $\mathcal{O}$.*

Proof. Let $K \subset \mathbb{R}^{m+1}$ be compact such that $K \cap \partial\mathbb{R}^{m+1}_+ \subset \mathcal{O}$ and suppose that $w \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ satisfies $v|_{\mathbb{R}^{m+1}_+\setminus K} = w|_{\mathbb{R}^{m+1}_+\setminus K}$. We need to show that $E^\beta(v) \le E^\beta(w)$. Define $\tilde{u} = Tw$ and let $\tilde{v}$ be a minimal harmonic map with $T\tilde{v} = \tilde{u}$, the existence of which follows from Lemma 6.0.0.1. We will show that for some compact set $\tilde{K} \subset \mathcal{O}$, we have $u|_{\mathcal{O}\setminus\tilde{K}} = \tilde{u}|_{\mathcal{O}\setminus\tilde{K}}$. Then, since v and $\tilde{v}$ are minimal harmonic maps and u minimises I^β in the sense of definition 6.0.0.1, we have

$$E^\beta(v) = I^\beta(u) \le I^\beta(\tilde{u}) = E^\beta(\tilde{v}) \le E^\beta(w) \tag{6.0.37}$$

as required. Thus we proceed to prove the assertion $u|_{\mathcal{O}\setminus\tilde{K}} = \tilde{u}|_{\mathcal{O}\setminus\tilde{K}}$.

By assumption $\mathcal{O}$ is open in $\partial\mathbb{R}^{m+1}_+$ and $K_m := K \cap \partial\mathbb{R}^{m+1}_+ \subset \mathcal{O}$. Furthermore, $K_m \subset \mathcal{O}$ is compact. Hence $\mathrm{dist}^m(K_m; \partial\mathcal{O}) > 0$ where dist^m is the distance in $\mathbb{R}^m \times \{0\}$. Consequently we can choose an open set $\tilde{\mathcal{O}} \subset \mathcal{O}$ with $K_m \subset \tilde{\mathcal{O}} \subset \overline{\tilde{\mathcal{O}}} \subset \mathcal{O}$. Since K_m is closed and $\tilde{\mathcal{O}}$ is open we have $\mathrm{dist}^m(K_m; \partial\tilde{\mathcal{O}}) > 0$ as well.

Hence, since $K_m \subset \mathcal{O}$ by assumption, we have $\mathrm{dist}(\mathcal{O}\backslash\overline{\tilde{\mathcal{O}}}; K) := \kappa > 0$ where dist is the distance in $\overline{\mathbb{R}^{m+1}_+}$. Observe that, on $\mathcal{O}\backslash\overline{\tilde{\mathcal{O}}}$, T coincides with $\tilde{T} \circ I_2 \circ I_1$ where $I_1 : \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; \mathbb{R}^n) \to W^{1,2}_\beta(\mathcal{O}\backslash\overline{\tilde{\mathcal{O}}} \times (0,\kappa); \mathbb{R}^n)$ is the imbedding given by Lemma 2.2.1.1, $I_2 : W^{1,2}_\beta(\mathcal{O}\backslash\overline{\tilde{\mathcal{O}}} \times (0,\kappa); \mathbb{R}^n) \to W^{1,p}(\mathcal{O}\backslash\overline{\tilde{\mathcal{O}}} \times (0,\kappa); \mathbb{R}^n)$ is the imbedding given by either Lemma 2.2.1.3 or Lemma 2.2.1.4, for p depending on β, and $\tilde{T}$ is the trace operator $\tilde{T} : W^{1,p}(\mathcal{O}\backslash\overline{\tilde{\mathcal{O}}} \times (0,\kappa); \mathbb{R}^n) \to L^p(\mathcal{O}\backslash\overline{\tilde{\mathcal{O}}}; \mathbb{R}^n)$. It follows that

$$\int_{\mathcal{O}\backslash\tilde{\mathcal{O}}} |u - \tilde{u}|^p \mathrm{d}x = \int_{\mathcal{O}\backslash\tilde{\mathcal{O}}} |T(v-w)|^p \mathrm{d}x \leq C||v-w||^p_{W^{1,p}((\mathcal{O}\backslash\tilde{\mathcal{O}})\times(0,\kappa);\mathbb{R}^n)} = 0$$

since $v = w$ in $\mathcal{O}\backslash\overline{\tilde{\mathcal{O}}} \times (0,\kappa)$. Hence (6.0.37) holds, with $\tilde{K} = \overline{\tilde{\mathcal{O}}}$ and the proof is complete. □

As a consequence of the preceding lemma, we can use the partial regularity theory, developed in Chapters 3 and 5, for minimisers of E^β relative to $\mathcal{O}$ to conclude corresponding partial regularity for minimisers of I^β.

Theorem 6.0.0.2. *Suppose $u \in T(\dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N))$ minimises I^β. Then there exists a $\gamma \in (0,1)$ and a relatively closed set $\Sigma \subset \mathcal{O}$ with $\mathcal{H}^{m+\beta-1}(\Sigma) = 0$ such that $u \in C^{1,\gamma}(\mathcal{O}\backslash\Sigma; N)$.*

Proof. Lemma 6.0.0.1 yields a minimal harmonic map $v \in \dot{W}^{1,2}_\beta(\mathbb{R}^{m+1}_+; N)$ with $Tv = u$. As u is a minimiser of I^β, Lemma 6.0.0.4 implies v is a minimiser of E^β relative to $\mathcal{O}$. Thus we may apply Theorem 5.6.0.1 to conclude the result. □

Chapter 7

Further Work

Our main results are a partial regularity theorem for minimisers v of E^β relative to $\mathcal{O}$, see Chapter 5 Theorem 5.6.0.1, and a partial regularity theorem for fractional harmonic maps u, see Chapter 6 Theorem 6.0.0.2. In particular, these theorems show that v and its derivatives $\partial_i v$, for $i = 1, \ldots, m$, are in $C^{0,\gamma}$ and $u \in C^{1,\gamma}$ away from their respective singular sets. In analogy with the theory of harmonic maps, we would like to show that fractional harmonic maps are smooth away from their singular set, or to extend our theory to hold for derivatives $D^{\tilde{\alpha}}u$, where $\tilde{\alpha}$ is a multi-index with $\tilde{\alpha}_{m+1} = 0$, of as higher order as possible. This will require extensions of the theory in Chapter 5. One such modification would be a substitute for the application of Lemma 3.6.0.3, which itself comes from the theory of harmonic maps [45], in establishing the L^∞ bound for the gradient of a minimiser. We would also need counterparts to the other lemmata in Chapter 5, with assumptions which encompass a general structure of the equations satisfied by the derivatives $D^{\tilde{\alpha}}v$ when $\tilde{\alpha}_{m+1} = 0$.

We have not, so far, considered the regularity of the derivatives of a minimiser v of E^β relative to $\mathcal{O}$ with respect to x_{m+1}, other than proving a bound for the gradient of v assuming its energy is sufficiently small. The theory, described in Chapter 4, for solutions to the linear Neumann type problem (2.4.5) suggests that it may be prudent to consider the quantity $x_{m+1}^\beta \partial_{m+1} v$ as well as $\partial_{m+1} v$. Moreover, information regarding higher order derivatives with respect to x_i, for $i = 1, \ldots, m$, may yield information about the derivatives with respect to x_{m+1}. Indeed, if we can, say, establish the boundedness of $\Delta' v$ in addition to ∇v then an analysis of the Euler-Lagrange equations for v together with the modified lemma of Morrey, Lemma 3.5.0.2, may yield information about the regularity of $x_{m+1}^\beta \partial_{m+1} v$.

We have seen in Chapter 3, Lemma 3.12.2.1, that the singular set of minimising fractional harmonic maps can be characterised very similarly to the singular set of harmonic maps. We may consider arguments to reduce the Hausdorff dimension of the singular set, analogous to Simon's refinement of the dimension reducing arguments of Federer, see [46] Chapter 3. The main difference between the situation considered in [46] and our own is that the bound for Hausdorff dimension of the singular set of minimising fractional harmonic maps depends on the parameter $\beta \in (-1,1)$ and is non-integer when $\beta \neq 0$. There are also geometric conditions in the theory of harmonic maps which reduce the potential dimension of the singular set. For instance, if the sectional curvature of N is non-positive then the singular set is always empty [44]. It would be interesting to investigate the possibility of such conditions lowering the dimension of the singular set of fractional harmonic maps.

Sometimes in regularity theories for harmonic maps, see [3] for example, the domain manifold is replaced (provided the metric is bounded and sufficiently regular) with Euclidean space and conclusions about regularity are established in this situation. The arguments are then modified accordingly for more general domain manifolds. The regularity theory we have developed for minimising fractional harmonic maps holds for open subsets of Euclidean space. It would be interesting to establish conditions which would allow the definition and analysis of fractional harmonic maps on more general Riemannian manifolds. For example, if $(M,\hat{g})$ is a manifold contained in the boundary of another manifold $(\tilde{M},\tilde{g})$ and if, in coordinates centred on $p \in M$, we have $c\tilde{g} \leq g \leq C\tilde{g}$ as tensors, where g is the metric defined by (3.0.1) in Chapter 3, then it may be possible to modify our theory to define and yield partial regularity for fractional harmonic maps on M.

A key milestone in proving the Hölder continuity of minimisers of E^β relative to $\mathcal{O}$ was the construction of comparison maps in Section 3.8. The remaining steps in our theory were not specific to minimising v. It may, therefore, be possible to consider the regularity of other critical points of I^β using some of the theory we have developed, provided we can find substitutes for the use of comparison maps and a connection with critical points of E^β. For instance, we could impose a stationarity condition, similar to the condition defining stationary harmonic maps, on critical points of I^β. As in [34], for the case $\beta = 0$, we may then hope that stationary critical points of I^β are linked with weakly stationary harmonic maps with respect to the Neumann type boundary condition (3.1.11). These maps would also be interesting to consider in their own right. It may be possible to adapt techniques used in the regularity theory for stationary harmonic maps, such as the moving frame method used by Hélein [26] and Bethuel [3], to stationary critical points of E^{β}. We may also try to extend Rivière and Struwe's approach to the regularity of stationary harmonic maps, investigating possible gains in regularity due to compensation phenomena [40].

We mentioned in Section 2.2.1 that when $m = 2$ we do not know if the embedding $\dot{W}^{1,2}_{\beta}(\mathbb{R}^{m+1}_{+};\mathbb{R}^{n}) \hookrightarrow W^{1,2}_{\beta}(\Omega;\mathbb{R}^{n})$, where $\Omega \subset \mathbb{R}^{m+1}_{+}$ is open, holds if $\beta \in (-1, -\frac{1}{3}]$. This prevents us from applying any of the theory for weighted Sobolev spaces or degenerate elliptic equations to solutions of $\mathrm{div}(x_{m+1}^{\beta}\nabla v) = 0$ and, as a result, our theory for Fractional Harmonic Maps does not hold in this case. It would be interesting to know if this is a consequence of our methodology or whether there is some other underlying reason that prevents our theory from holding in this situation.

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